PROGRESSIONS IN AREA MODELS

Lane Walker
@LaneWalker2 lhwalker1337@gmail.com SLIDES

Arrays and area models, frequently mentioned in the Common Core State Standards for Mathematics, are among many instructional tools that teachers may use to present (and students may use to explain) important mathematical concepts. As topics become more abstract in algebra, arrays and area models can provide powerful visualizations and connections for students who are already familiar with their application at the concrete levels, but may be confusing and frustrating to students who are not familiar. For this reason, teachers at the upper levels often forgo their use if a large number of their students have not had the exposure. Many of those students also struggle to recall relationships between perimeter and area learned in other contexts. The potentiality for cohesiveness in these models should be considered when designing curriculum vertically around best practices. Examples for the use of area models aligned with the standards at each grade level follow. In some (but not all) cases the standards themselves mention arrays or area models.

> "Students need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP7). Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions."[1]

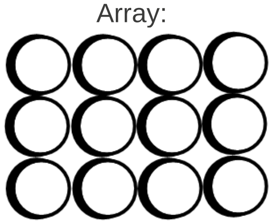
Instruction with area models begins with each square representing the unit 1.

Grades K-1

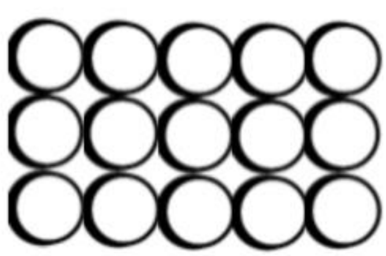
K.CC.B Count to tell the number of objects.

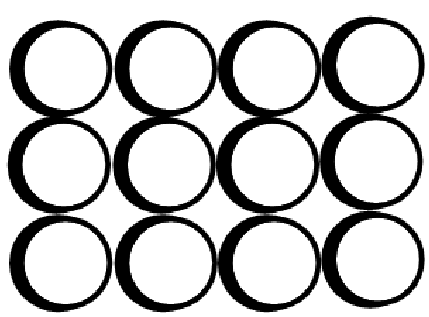
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a **rectangular array**, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Array:



6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies





1.OA.C.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten

How many more make 10?

How many more make 9? (eventually related to "completing the square in Algebra 1):

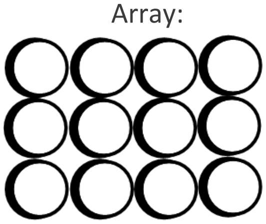
Connections with dyscalculia:
1) Students count the objects one-by-one, associating the objects with the verbal numbers
2) Students practice skip counting and/or subitizing by rearranging arrays in rows of 2 or 3
3) Counting on
4) Comparing numbers of objects
5) Making 10.

Grade 2

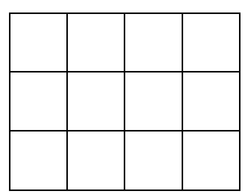
2.OA.C Work with equal groups of objects to gain foundations for multiplication.
3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
4. Use addition to find the total number of objects arranged in **rectangular arrays** with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.
2.G Reason with shapes and their attributes.

2G.2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Array:



Area Model:

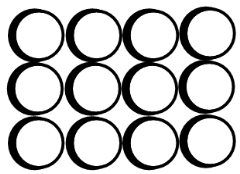


Grade 3

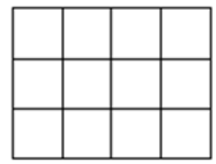
The introduction to the Grade 3 CCSS states (emphasis added), " Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, **arrays**, and **area models**; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division….Students **recognize area as an attribute of two-dimensional regions**. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that **rectangular arrays** can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle."

"Students might then find the areas of other rectangles. As previously stated, students can be taught to multiply length measurements to find the area of a rectangular region. But, in order that they make sense of these quantities (MP2), they first learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows.3.MD.7a This relies on the development of spatial structuring. "[3]

Array



Area Model



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3.OA.A Represent and solve problems involving multiplication and division.

1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.*

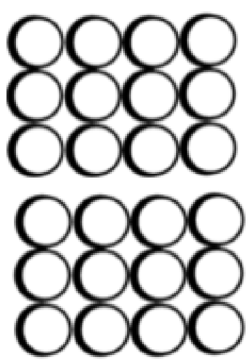
2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*

4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8x? = 48, 5 = □÷3, 6x6=?

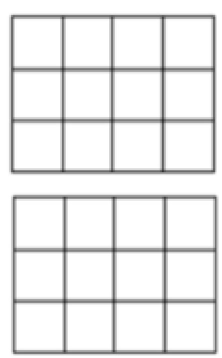
While area models are not explicitly mentioned in 3OA.A, their use would logically follow that of multiplication and lead to other topics such as 3MD.C.

2 x 12 = 24

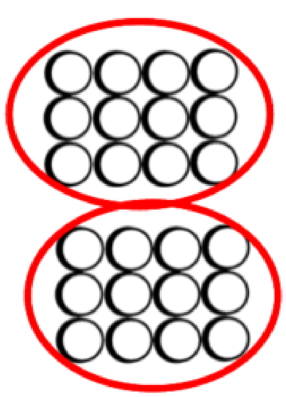
Array

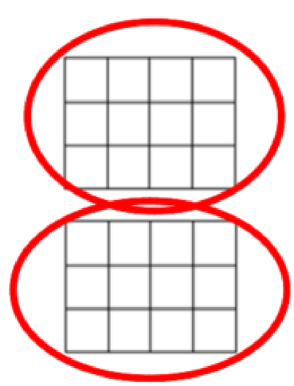


Area Model



24 ÷ 2 = 12

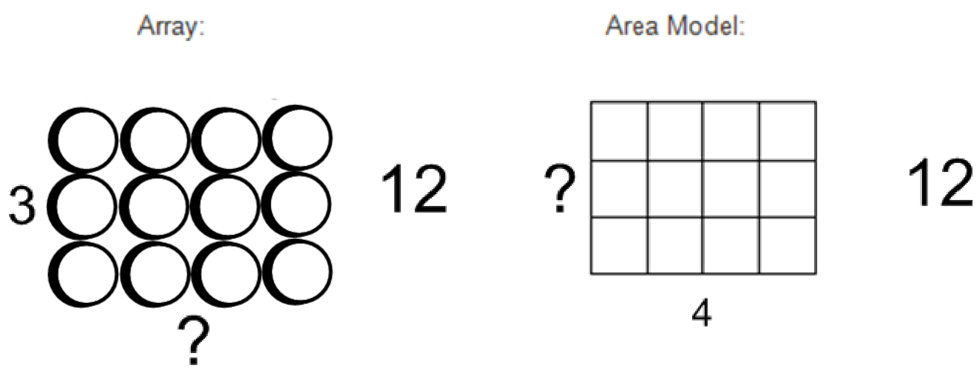




Connections with dyscalculia:
Students may struggle with the idea of "inverse operations" and "undoing" when solving equations in algebra if they do not understand the relationship between multiplication and division. Foundations for solving algebraic equations can be built with area models.

3.MD.C Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
5. Recognize area as an attribute of plane figures and understand concepts of area measurement.
a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.
6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).
7. Relate area to the operations of multiplication and addition.
a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

Supplying the product, this task involves finding the unknown:

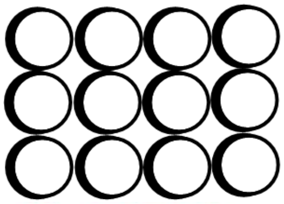


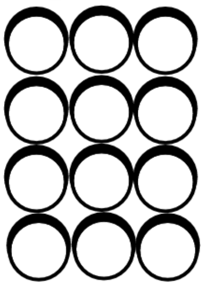
3.OA.A.3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, **arrays**, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.1
4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. *For example, determine the unknown number that makes the equation true in each of the equations*
$8 \times ? = 48$, $5 = \square \div 3$, $6 \times 6 = ?$.
6. Understand division as an unknown-factor problem. *For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8.*

3.OA.B Understand properties of multiplication and the relationship between multiplication and division.
5. Apply properties of operations as strategies to multiply and divide.2 *Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)*

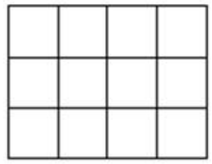
The distributive property:

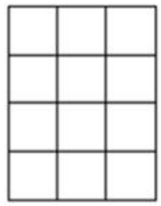
Array:





Area Model:

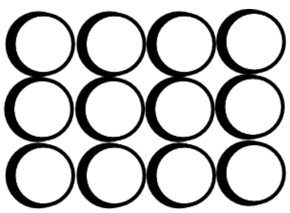


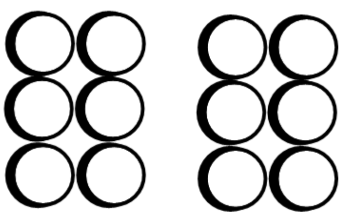


Connection with dyscalculia: Students with dyscalculia may not find the commutative property intuitive. It may be helpful for them to make an array or area model and then rotate it.

The associative property:

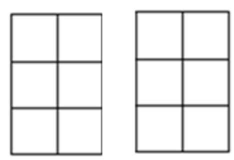
array



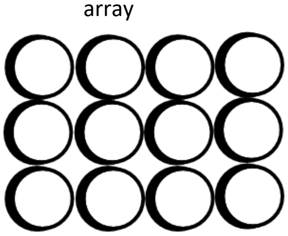


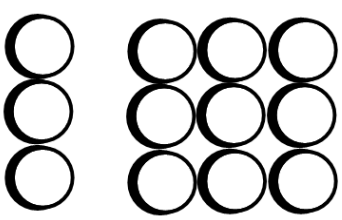
2 x 3 x 2

area model



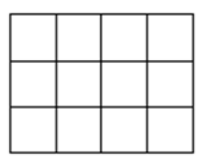
array

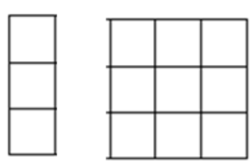




3 x (1 + 3)

area model

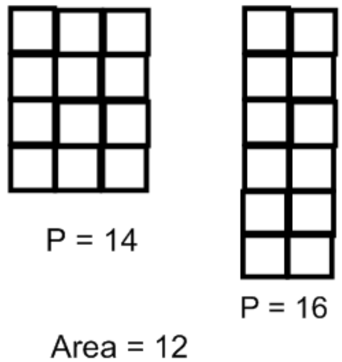


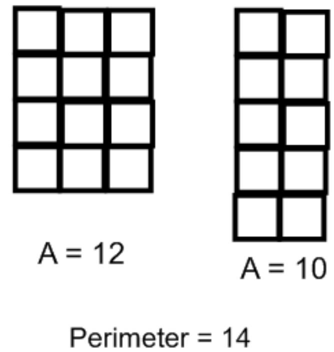


3 x (1 + 3)

Physical tiles can enable students to manipulate them into rectangles with the same area and different perimeters and rectangles with the same perimeters but different areas.

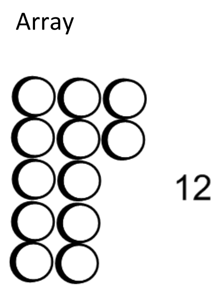
3.MD.D.8 Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

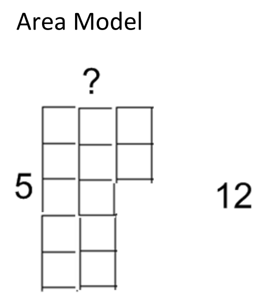




Grade 4

Working with area models and arrays extensively in Grade 3, transitions well into finding factors of numbers. The use of manipulatives could facilitate explorations as students determine whether particular side lengths produce a complete rectangle with a given number of blocks. The *total* length of an edge of an area models is a *factor* of the *total* area. Five is not a factor of 12:





4.OA.B Gain familiarity with factors and multiples.
4. Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

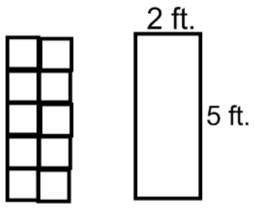
While 4.OA.B does not mention using arrays and area models to find factors, doing so would add a concrete perspective that connects to and reviews prior experiences. Each of these explorations provide opportunities for transfer students to become familiar.

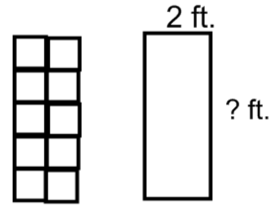
It is important to connect the area models used with the real life applications in 4.MD.A.3 as not all students will do this intuitively.

4.MD.A.3 Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.

How many square feet of carpet would be required to carpet the closet floor drawn below?

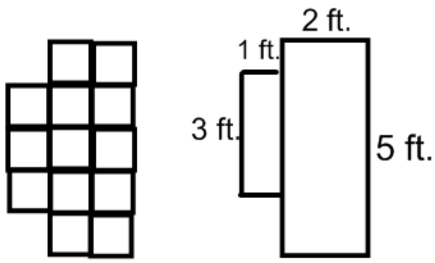
If the area of the closet floor drawn below is 10 ft^2, what is the missing dimension?





We cannot say, "We just multiply to find the area" because total area frequently requires addition. To find perimeter, students can double the length and width, instead of adding four sides. So we cannot say, "For the perimeter, we always just add."

How many square feet of carpet would be required to carpet the foyer drawn below?



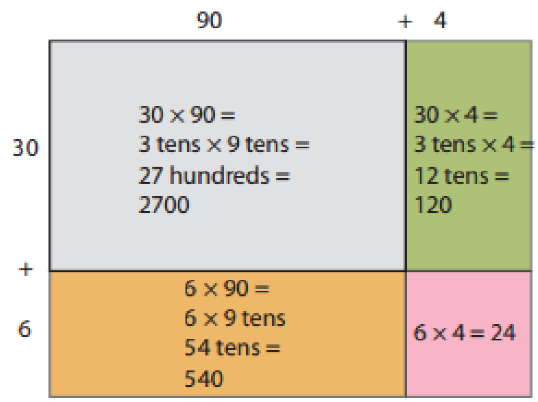
4.NBT.B Use place value understanding and properties of operations to perform multi-digit arithmetic.
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, **rectangular arrays, and/or area models**.

Work with real-life models along with area models connects to the next step in multiplication where the unit grids are not shown.

4.NBT.B
5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, **rectangular arrays, and/or area models.**

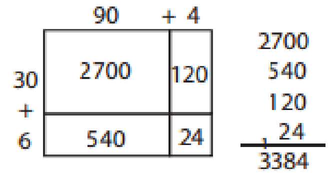
The following diagram was published by Karen C. Fuson, Northwestern University, and Sybilla Beckmann, University of Georgia NCSM JOURNAL • FALL/WINTER 2012-2013 *Standard Algorithms in the Common Core State Standards* as adapted from the CCSS NBT Progressions.[4]

Array/area drawing for 36 × 94

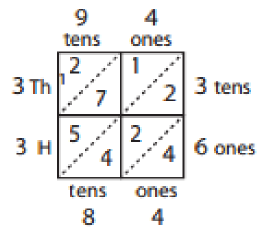


$$36 \times 94 = (30 + 6) \times (90 + 4)$$
$$= 30\times90 + 30\times4 + 6\times90 + 6\times4$$

Area Method F:



Lattice Method G:

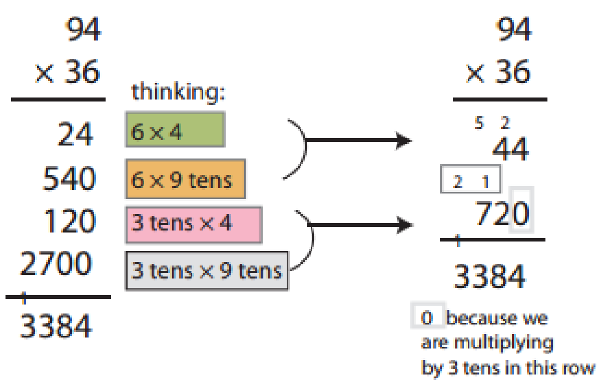


Method D:

Showing the partial products

Method E:

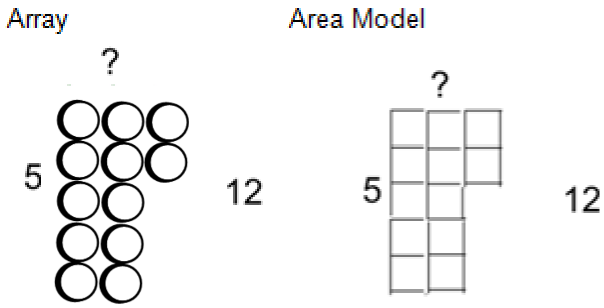
Recording the carries below for correct place value placement

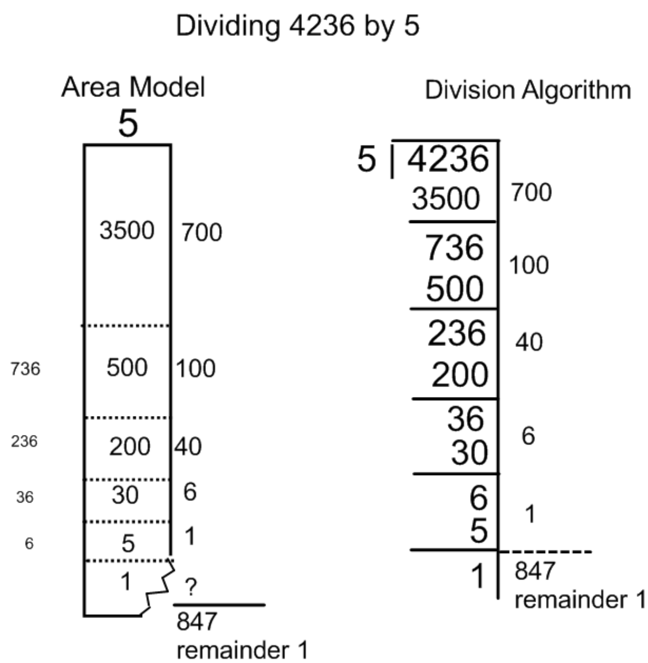


Written Methods D and E are shown from right to left, but could go from left to right. In Method E, digits that represent newly composed tens and hundreds in the partial products are written below the line intead of above 94. This way, the 1 from 30 × 4 = 120 is placed correctly in the hundreds place and the digit 2 from 30 × 90 = 2700 is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 in the ones place of the second line of Method E is there because the whole line of digits is produced by multiplying by 30 (not 3).

Leaving out the unit grids leads students to begin to visualize them in their heads. It is equally important to frequently revisit problems with the grids so that the connection between the area and the number of unit squares not be lost over time or misunderstood by students who have not seen area models before. Maintaining the grid connection is also important for introducing variable lengths in upper grades.

4.NBT.B.6. Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, **rectangular arrays, and/or area models**

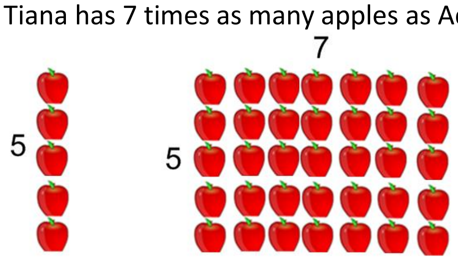




Comparison equations are considered a higher order reasoning beyond multiplying two numbers together.[5] The language "as many as" and "as much as" are important expressions in CCSS-M, related to ratios, proportional reasoning, and multiplying fractions:

4.OA.A.1 Use the four operations with whole numbers to solve problems.
1. Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons equations.as multiplication

Tiana has 7 times as many apples as Aqib. If Aqib has five apples, how many does Tiana have?



Grade 5

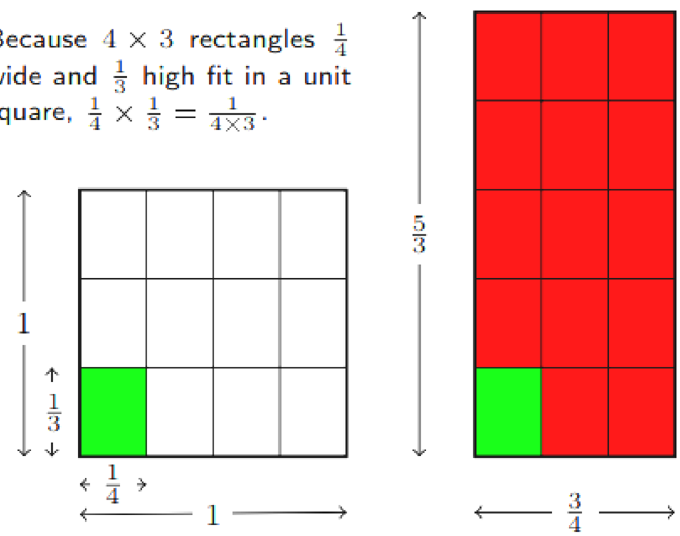
Working with fractions is a major focus in the 5th grade standards. While tape diagrams are an integral part of conceptually understanding fractions, area diagrams can show the effects of multiplication in a way that is connected to multiplication with integers.

5.NF.B.4.B. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Progressions: 3–5 Number and Operations—Fractions[6]

Using an area model to show that $\frac{3}{4} \times \frac{5}{3} = \frac{3\times5}{4\times3}$

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high fit in a unit square, $\frac{1}{4} \times \frac{1}{3} = \frac{1}{4\times3}$.

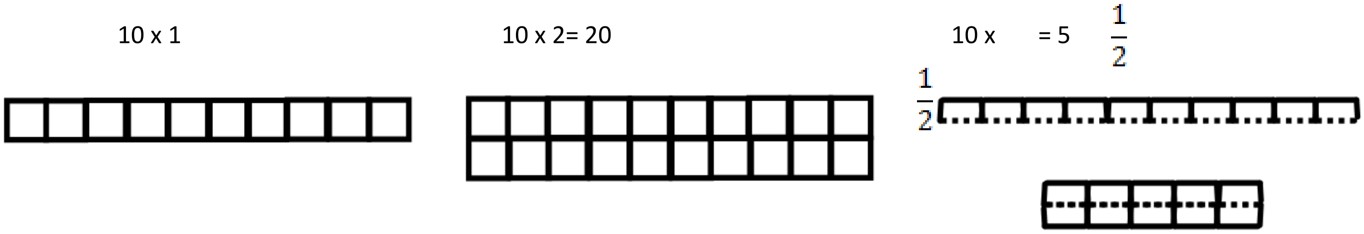


The rectangle of width $\frac{3}{4}$ and height $\frac{5}{3}$ is tiled with 3×5 rectangles of area $\frac{1}{4\times3}$, so has area $\frac{3\times5}{4\times3}$.

5.NF.B.5. Interpret multiplication as scaling (resizing), by:

a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication. "Ten times as much…"

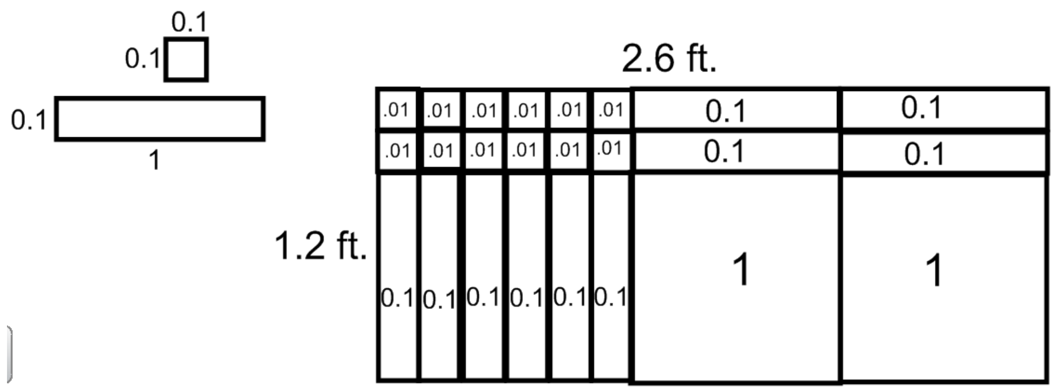
b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence *a*/*b* =(*n*×*a*)/(*n*×*b*) to the effect of multiplying *a*/*b* by 1.



In previous examples, a square has represented one square unit. However, a 1x1 square can be understood to represent an even integer power of 10.

5. NBT.B Perform operations with multi-digit whole numbers and with decimals to hundredths.
7. Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

2.6 x 1.2

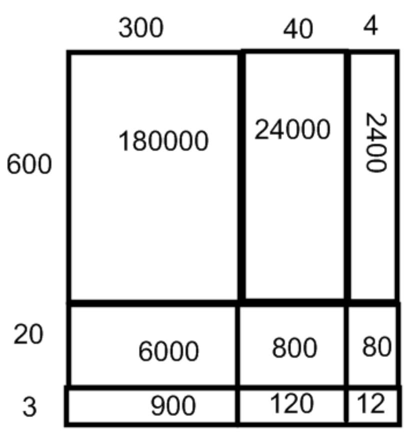


However, with good understanding of place value it might be easier to understand this model:

34.4 x 6.23

34.4 x 6.23 = 344 ÷ 10 x 623 ÷ 100

or 344 x 623 ÷ 1000

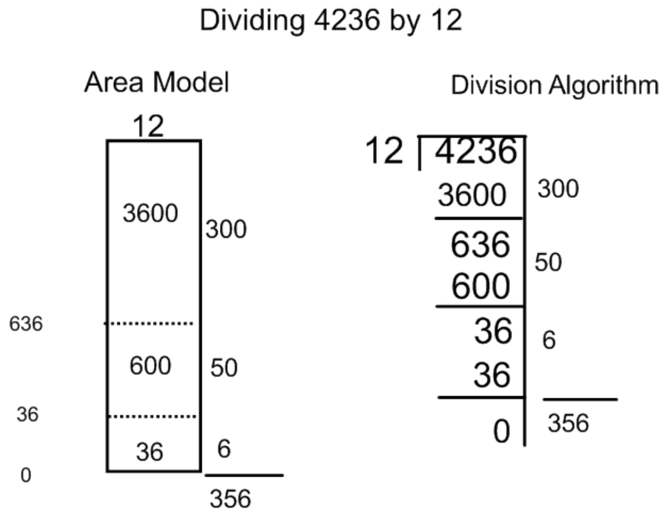


180,000
24,000
2,400
6,000
800
80
900
120
12

214,312

34.4 x 6.23 = 214.312

5.NBT.B.6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

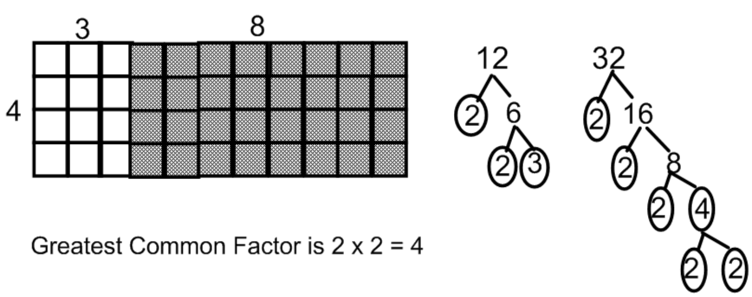


Grade 6

In the 6th grade standards, there are fewer opportunities where area diagrams are traditionally used. However, greatest common factors and least common multiples afford opportunities to productively use the models, not just for visualization, but also to keep topics connected for overall coherence, for review, and to give students transferring from other schools an opportunity to develop understanding of the models.

6.NS.B.4 Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*

> Application: You need to package 12 cartons of white milk with 32 cartons of chocolate milk in a rectangular crate and you want all of the same kind in each row (or column). Use an area model to show how you could do that, assuming the bases of the cartons are square.
>
> .

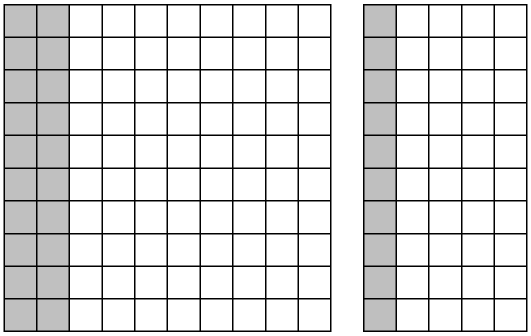


Distributive property: 4(3 + 8) = 44

Area diagrams can assist in understanding of a percent as a rate per 100 for obvious cases. Then students can move onto ratio and rate reasoning using tables, tape diagrams, etc. for less intuitive cases.

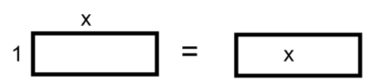
6.RP.A.3 **Understand ratio concepts and use ratio reasoning to solve problems.**
3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

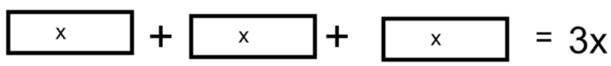
20% is a rate of 20/100 which is is equivalent to a rate of 10/50

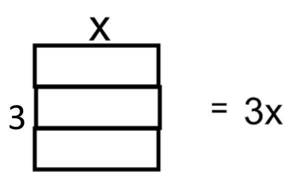


6.EE.A.3 Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y*.

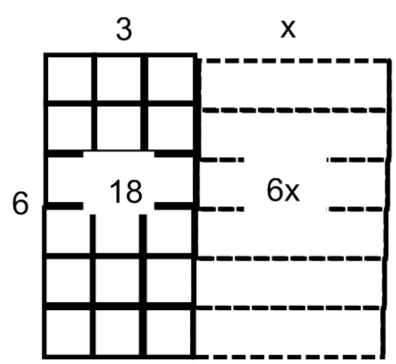
x + x + x = 3x

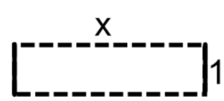






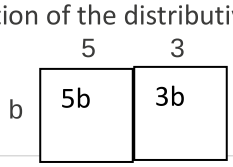
6 (3 + x) = 18 + 6x



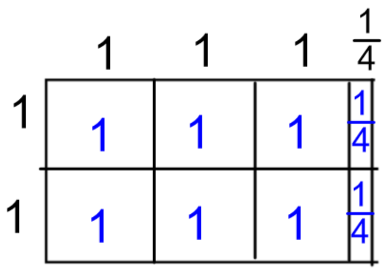


"The distributive law is of fundamental importance. Collecting like terms, e.g., 5b + 3b= (5 + 3)b = 8b, should be seen as an application of the distributive law, not as a separate method."[7]

b(5 + 3)

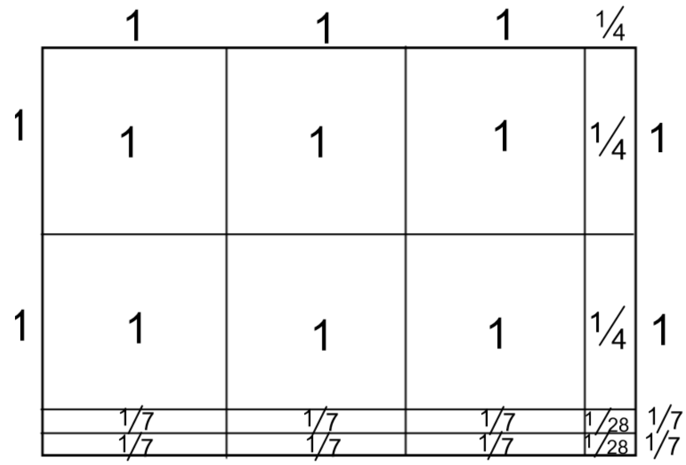


Mixed number multiplication



$2 \times 3\frac{1}{4} = 6\ \frac{2}{4}$

$2\,(3 + \frac{1}{4}) = 6\ \frac{2}{4}$



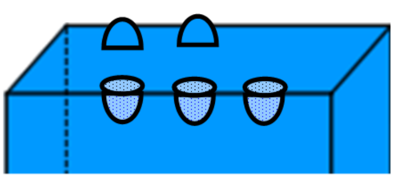
$2\frac{2}{7} \times 3\frac{1}{4}$

$(2 + \frac{2}{7})\,(3 + \frac{1}{4}) = 6 + \frac{2}{4} + \frac{6}{7} + \frac{2}{28}$

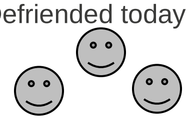
$= 6 + \frac{40}{28}$

$= 7\,\frac{12}{28}$

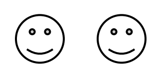
Positive and negative integers can best be visualized different ways by different students. Below is an example of hallows and mounds, representing positive and negative integers.[8] A similar construction could be made from classroom clay to show -3 + 2. Social media affords another way to visualize that is familiar and important to many students.



Defriended today

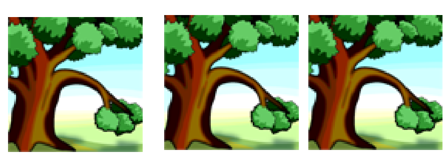


Friends added today

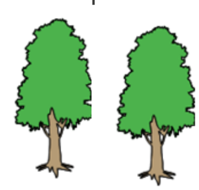


Kris has one fewer friend on facebook today than he had yesterday.

Trees Dying



Trees planted



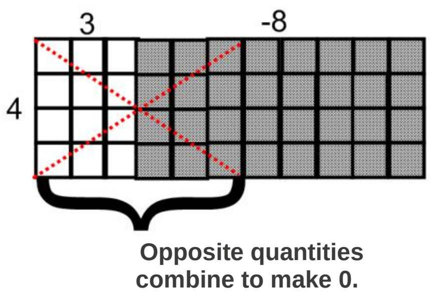
The meadow is losing one more tree than it is gaining.

6.NS.C Apply and extend previous understandings of numbers to the system of rational numbers.

5. Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts,explaining the meaning of 0 in each situation.

7.NS. A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
1.d. Apply properties of operations as strategies to add and subtractrational numbers.
2. c. Apply properties of operations as strategies to multiply and divide rational numbers.

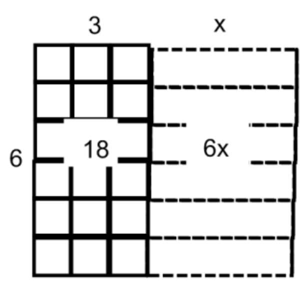
Illustrating operations with negative numbers in area models is challenging because area cannot be negative. However, showing area that "needs to be subtracted" helps students conceptualize and connect operations on integers and fractions.

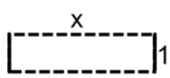


$4(3-8) = 4x3+4x{}^{-}8 = 4x3+4x{}^{-}3 + 4x{}^{-}5$

7.NS.A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide
a. Describe situations in which opposite quantities combine to make 0. *For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged.*
b. Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.

Operations including an unknown quantity follow the same strategies:



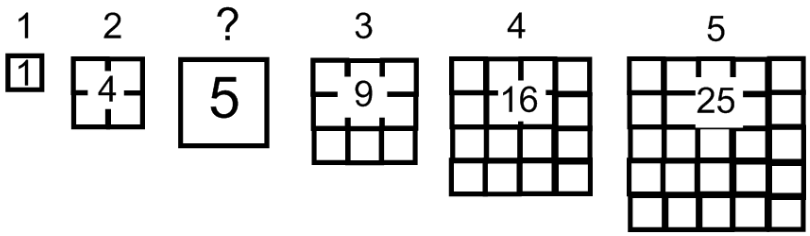


$6(3-x) = 18-6x$

Grade 8

8.EE.A Work with radicals and integer exponents.

2. Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that √2 is irrational.

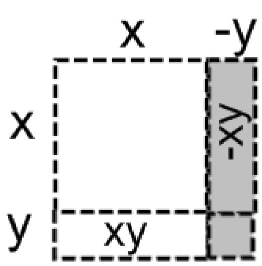


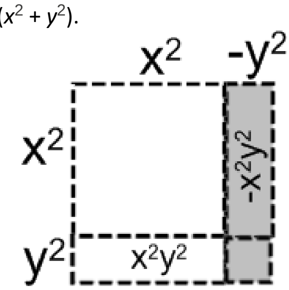
Pattern: $(n+1)^2 = n^2 + (n + 1) + n = n^2 + 2n + 1$

High School Algebra

A.SSE.A Interpret the structure of expressions

2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.





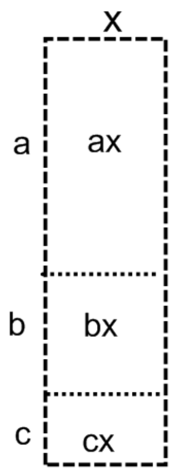
$(x - y)(x + y) = x^2 - y^2$

$(x^2 - y^2)(x^2 + y^2) = x^4 - y^4$

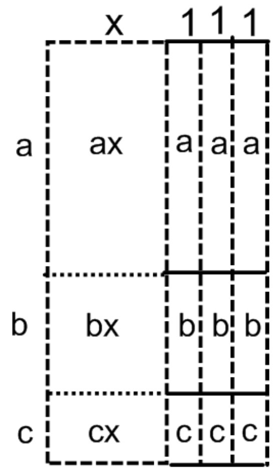
A-APR.A Perform arithmetic operations on polynomials

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

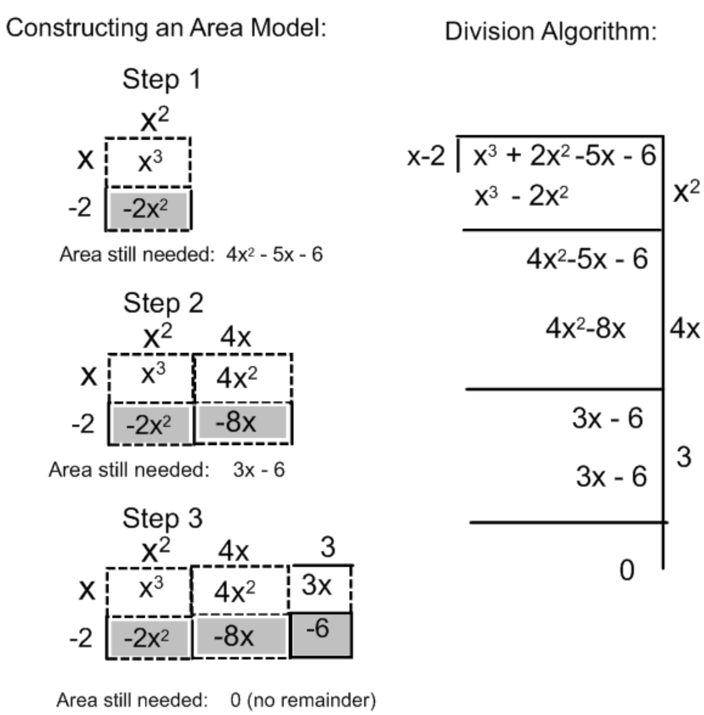
$x(a + b + c) = ax + bx + cx$



$(x + 3)(a + b + c)$



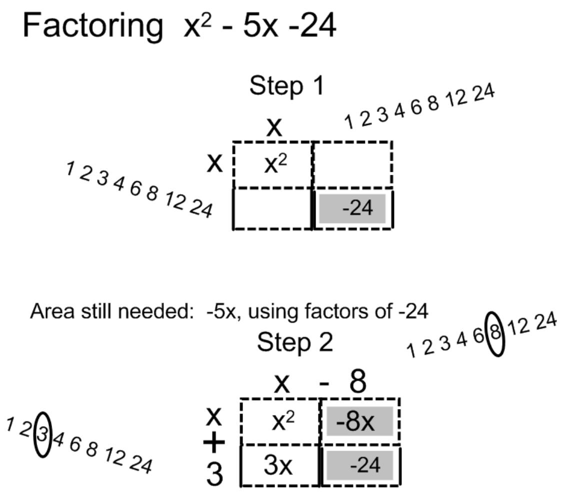
Dividing $x^3 + 2x^2 - 5x - 6$ by $x - 2$



A.SSE.B Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

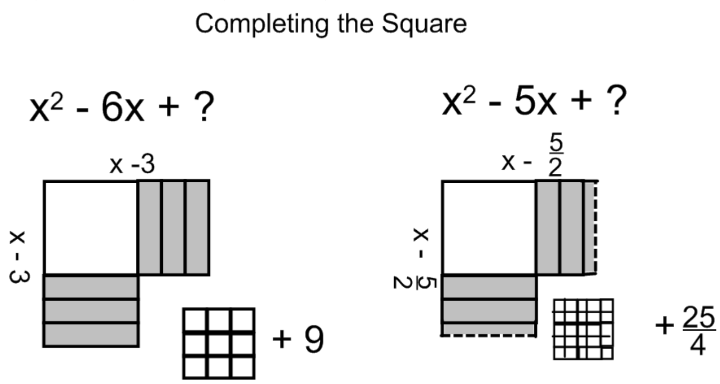
a. Factor a quadratic expression to reveal the zeros of the function it defines.



A.SSE.B Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.



[1]Draft K-6 progression: Geometry, The Common Core Standards Writing Team, 23 June 2012

[2]Draft progression: The Kindergarten Counting and Cardinality Progression and the K–5 Operations and Algebraic Thinking, The Common Core Standards Writing Team, 29 May 2011

[3]Draft progression: K–5 Geometric Measurement, The Common Core Standards Writing Team, 23 June 2012

[4]Draft progression: K-5 Number and Operations in Base 10, 21 April 2012

[5]Draft progression: Counting and Cardinality and Operations and Algebraic Thinking (K-2), 22 April 2011

[6]Draft progression: 3-5 Number and Operations -- Fractions, The Common Core Standards Writing Team, 12 August 2011

[7]Draft progression: 6-8 Expressions and Equations, The Common Core Standards Writing Team, 22 April 2011

[8]Sawyer, W.W. (1964), *Vision in Elementary Mathematics* Penguin Books Ltd., Middlesex, England