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Graph Theory and Research Experience:

A Nontraditional, Advanced but Accessible Elective Course

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The Game of Sprouts

- The Game of Sprouts begins with *n* distinct dots (known as **vertices or nodes**) placed at random on a page. A move consists of drawing an arc (called an **edge**) from one dot either to itself or to another dot, and placing a new dot somewhere on the arc. Only two rules must be observed:

  (1) The edge cannot cross itself, a previously drawn edge, or pass through a vertex.

  (2) No vertex can have more than three edges emanating from it.

- Players take turns drawing arcs. The winner is the last player able to draw an edge.

- If a game of Sprouts breaks out with *n* initial vertices, what can be said about the final graph resulting from the play of the game? How many vertices will it have? How many edges? How many regions will the final graph have? Is it possible to play an infinite game? Consider other generalizations you can make about the game.

Teaching Mathematics: The Rules of the Game

Dan Kennedy, Ph.D., Mathematics Professor, Author

The rules of that game are simple: we, the teachers show them what to do and how to do it; we let them practice at it for a while, and then we give them a test to see how closely they can match what we did.

What we contribute to this game is called "teaching," what they contribute is called "learning," and the game is won or lost for both of us on test day.

Ironically, thinking is not only absent from this process, but in a curious way actually counterproductive to the goals of the game.

Teaching Mathematics: The Rules of the Game

Dan Kennedy, Ph.D., Mathematics Professor, Author

Thinking takes time. Thinking comes into play precisely when you cannot do something "without thinking." You can do something without thinking if you really know how to do it well.

If your students can do something really well, then they have been very well prepared. Therefore, if both you and your students have done your jobs perfectly, they will proceed through your test without thinking.

Teaching Mathematics: The Rules of the Game

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If you want your students to think on your test, then you will have to give them a question for which they have not been fully prepared.

If they succeed, fine; in the more likely event that they do not, then they will rightfully complain about not being fully prepared.

You and the student will have both failed to uphold your respective ends of the contract that your test was designed to validate, because thinking will have gotten in the way of the game.

Considering how we mathematicians value thinking, it is a wonder that we got ourselves into this mess at all.

How do we do mathematics?

by practicing and remembering

OR

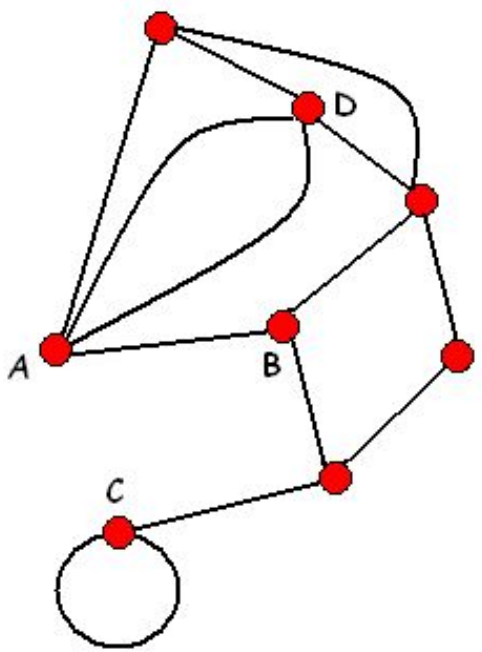
by understanding and thinking

Fundamentals of Graphs

- A **graph** consists of a finite non-empty collection of **vertices** and a finite collection of **edges** joining those vertices.

- Two vertices are **adjacent** if they have a joining edge. An edge joining two vertices is said to be **incident** to its endpoints.

- The **degree** of a vertex *v* is the number of edges which are incident to *v*.



Early Results and Introduction to Proof

Theorem 1: Let G be a simple connected planar graph with $V \geq 3$, then $E \leq 3(V-2)$.

Proof: G is a simple connected planar graph, so $\sum \deg(f_i) = 2E$.

Since the graph is simple, all faces must be of degree 3 or more (there are no loops or multiple edges), so $\sum \deg(f_i) \geq 3F$.

Consequently, $F \leq \frac{2}{3}E$.

Also, $V - E + F = 2$, so $V - E + \frac{2}{3}E \geq 2$ and $E \leq 3(V-2)$.

Early Results and Introduction to Proof

Theorem 2: Let G be a simple planar connected graph.
Then G has at least one vertex of degree 5 or less.

Proof: We proceed by contradiction. Suppose all of the vertices of G have degree 6 or more, so $\sum \deg v_i \geq 6V$. But, by the handshaking lemma, $\sum \deg v_i = 2E$. So $E \leq 3V$.

But G is a simple planar connected graph, so $E \leq 3(V-2)$.

This contradiction shows it is not possible for all of the vertices to be of degree 6 or more, so at least one must be of degree 5 or less.

Course Work

- Group Work problems – given a class period to work on problems in groups of 3-4. Solutions turned in at end of class meeting.
- Individual assignments – given approximately a week to work on with a partner outside of class.
- One in-class assessment – 90 minutes; focused more on core competencies of Graph Theory.
- Final research project – work in small groups for 2-3 weeks. Paper and presentation given during exam period.

Research Experience

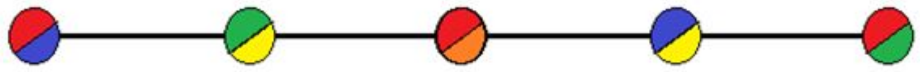
Star Coloring

A **star coloring** of a graph G is a vertex coloring of G for which every path on four vertices uses more than two colors (and such that, as usual, adjacent vertices are different colors). The **star chromatic number** of G, denoted $\chi_S(G)$, is the least number of colors required to star color G. For this problem you will explore star colorings for various graphs and graph families. One potential question you could consider would be determining the conditions under which $\chi(G) = \chi_S(G)$.

Research Experience

t-tone k-colorings of a Graph

Given a graph G, we create a t-tone coloring of each vertex (chosen from k colors) so that any two vertices a distance d apart share fewer than d common colors. The minimum integer k such that a graph has a t-tone k-coloring is known as the t-tone chromatic number. A 2-tone 5-coloring of the path P_5 is shown below, illustrating the 2-tone chromatic number $\tau_t(P_5) = 5$.



For this problem you will explore the t-tone chromatic number for various graphs and graph families. One potential question to explore would be determining for what graphs can the 3-tone chromatic number be found.

Grading Approach:
Standards-Based Grading

- Rather than given percentage scores on assignments, students are scored individually on certain standards:
  - **Core competencies:** Implementing Processes and Algorithms, Using Definitions, Familiarity with Graph Properties, Using Theorems and Other Previous Results
  - **Advanced Skills:** Communicating, Synthesizing, Investigating, Developing and Testing Conjectures, Constructing Logical Arguments
- Each standard is assessed multiple times, and the more recent scores weigh more heavily than early scores to determine final course grade.
- Goal is to focus on mastery of the subject and not penalize students for challenges early on.

Thank You and Questions!

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