Bringing Geometry to Precalculus

Using Matrices to Transform Plane Figures

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Today's agenda

- What **motivated** us to present this talk? [4 min]
- **Sneak preview** of transformations of plane figures [4 min]
- ***Making connections*** between numerical/algebraic concepts and geometric transformations:
  - **Review of key concepts** [10 min]
  - **"Cracking the code"** [12 min]
- ***Working through examples*** of **using matrices to transform geometric figures** in the plane [22 min]
- **Questions and discussion** [8 min]

AP PRECALCULUS

UNIT 4

Functions Involving Parameters, Vectors, and Matrices

Additional Topics Available to Schools
(not included on AP Precalculus Exam)

4.8 Vectors

4.10 Matrices

4.12 Linear Transformations and Matrices

4.13 Matrices as Functions

Linking representations

Linking graphical, numerical, and algebraic modes of thought created **analytic geometry**. This, in turn, led to the development of **calculus**.

Translating across representations is a key student practice in **AP® Precalculus**. Students present in verbal, graphical, numerical, and algebraic modes to make their ***thinking*** seen and heard by others, and thus develop their **power of visualization**.

AP® Precalculus Mathematical Practices

Procedural and Symbolic Fluency

Algebraically ***manipulate*** functions, equations, & expressions.

Multiple Representations

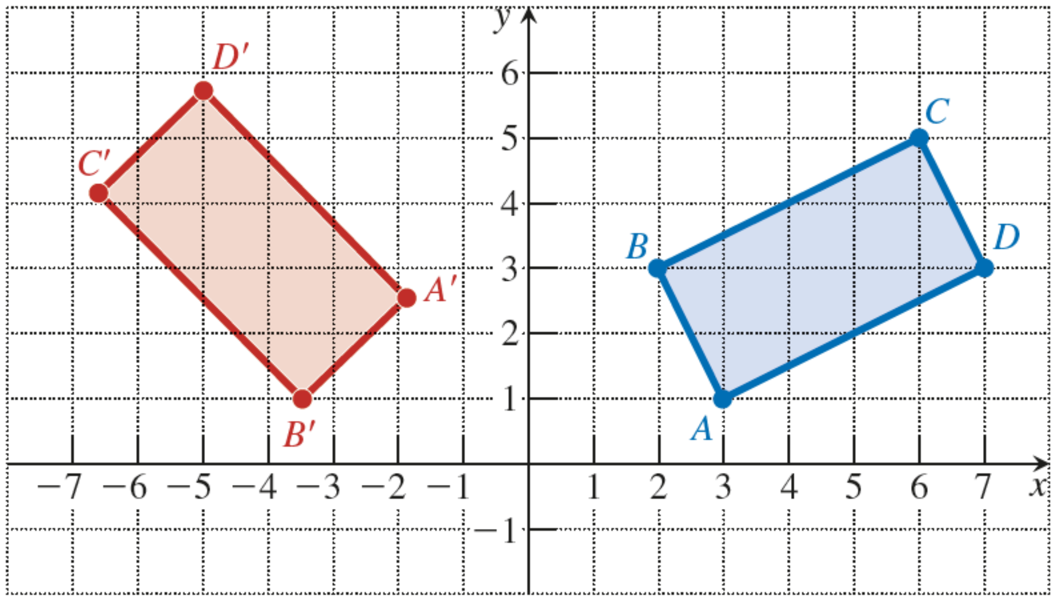
Translate mathematical information between representations.

Communication and Reasoning

Communicate with precise language & provide rationales for conclusions.

Focus on student thinking, inquiry & communication

Sneak preview of geometric transformations



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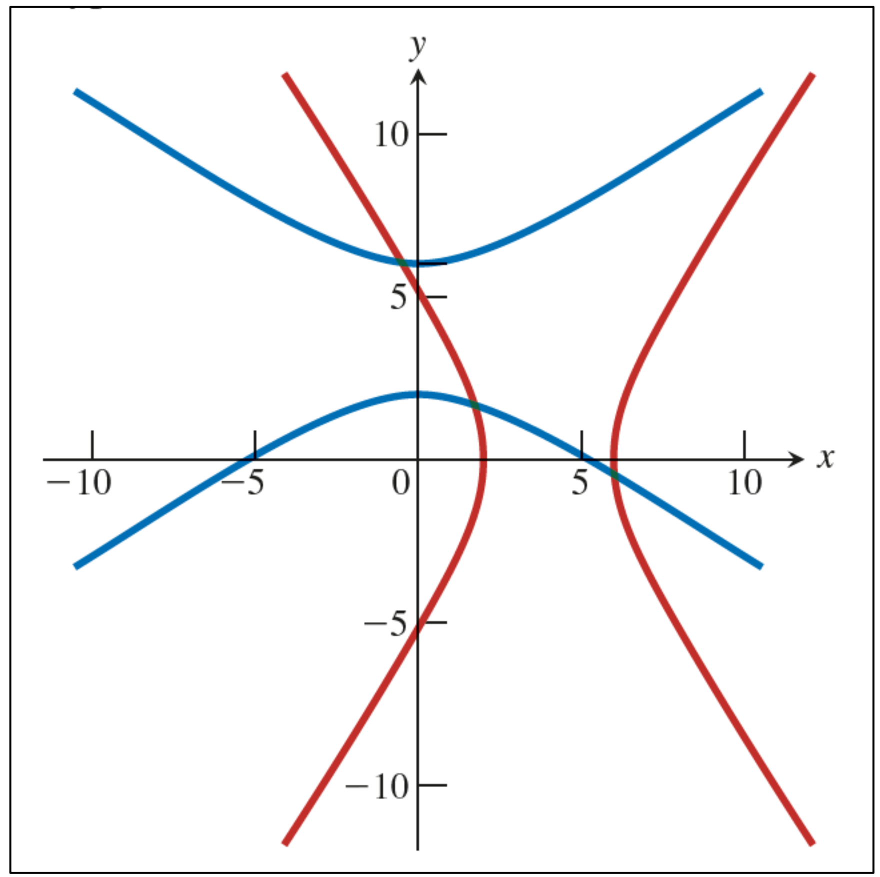
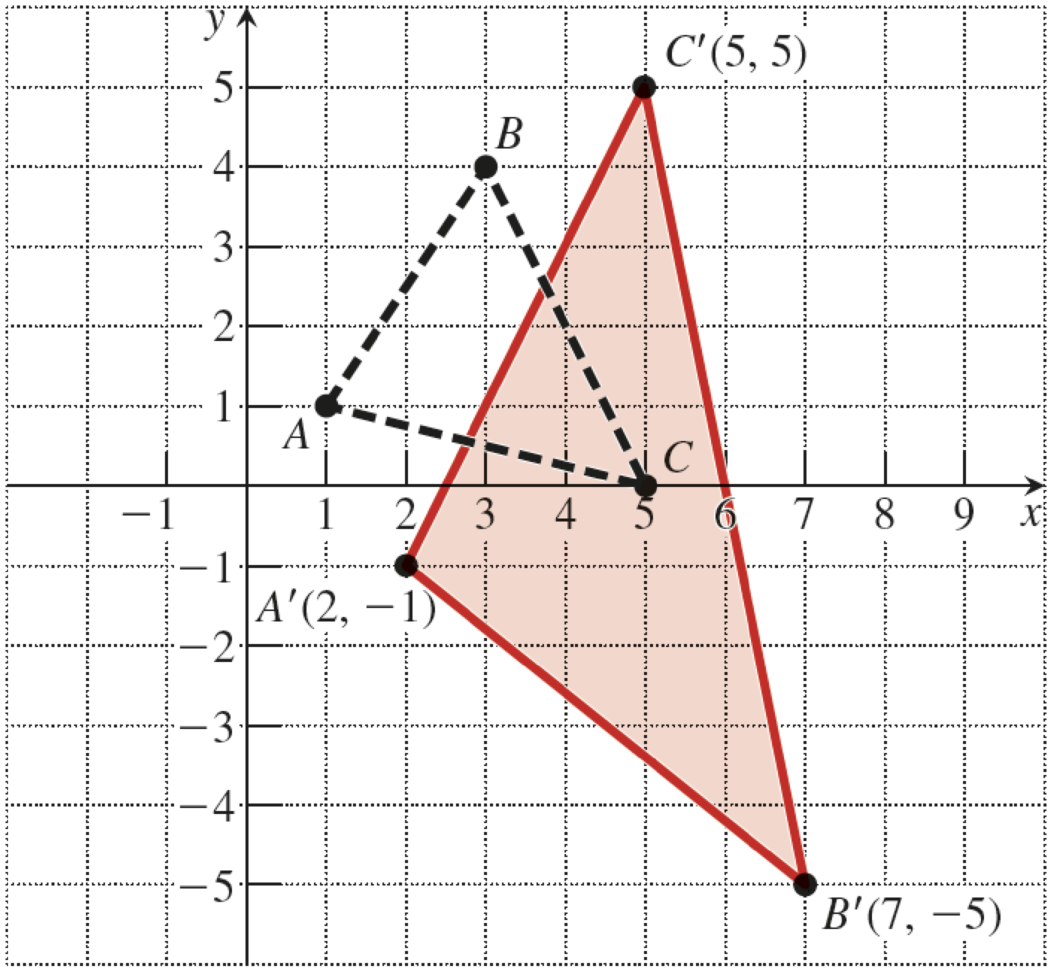
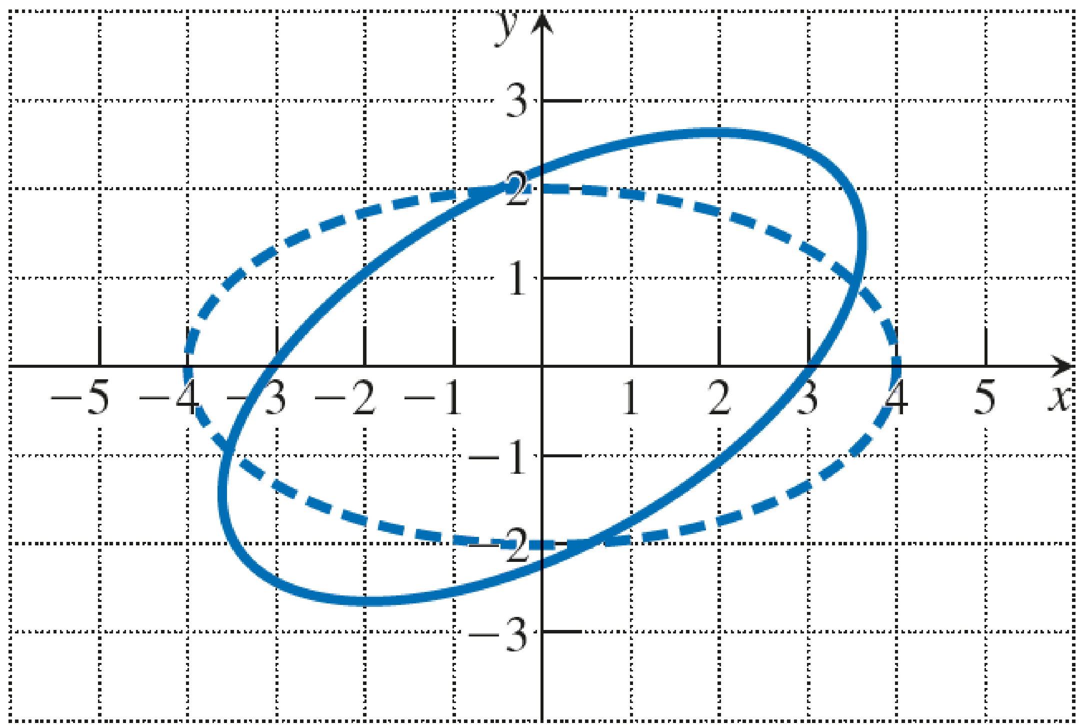


Sneak preview of geometric transformations



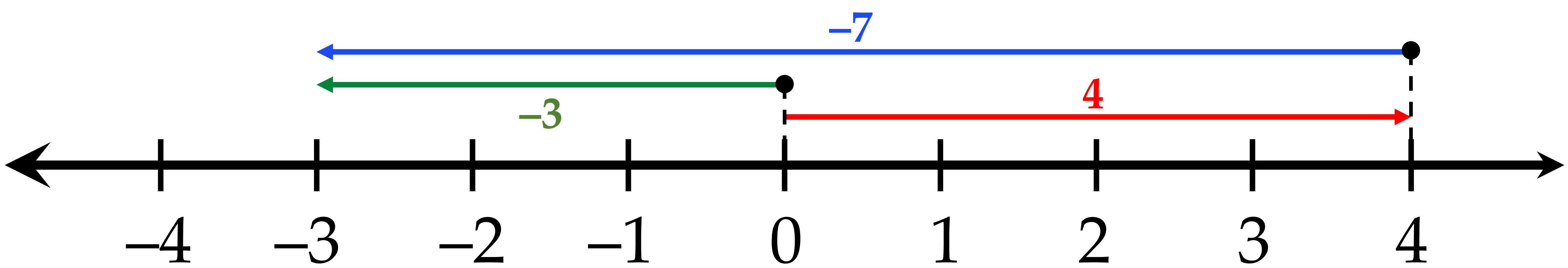


Review of 1 dimensional (1D) concepts

- One (1) dimensional space—the **number line**—connects real numbers to points on a geometric line.
- To visualize a computation like **4 + –7 = –3**, we use **arrows** (which represent numbers as 1D **vectors**!)
- ***Transformations*** (stretching, shrinking, or reflecting) of these vectors can be done using **scalar multiplication** of these 1D vectors, such as: **–3 · (1.5) = –4.5**

Vector addition in 1 dimension

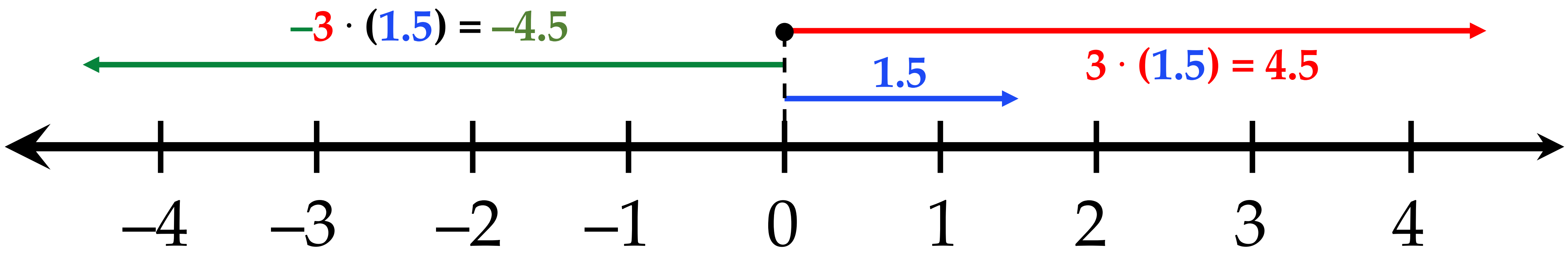
Vector representation of $4 + -7 = -3$



Scalar multiplication in 1 dimension

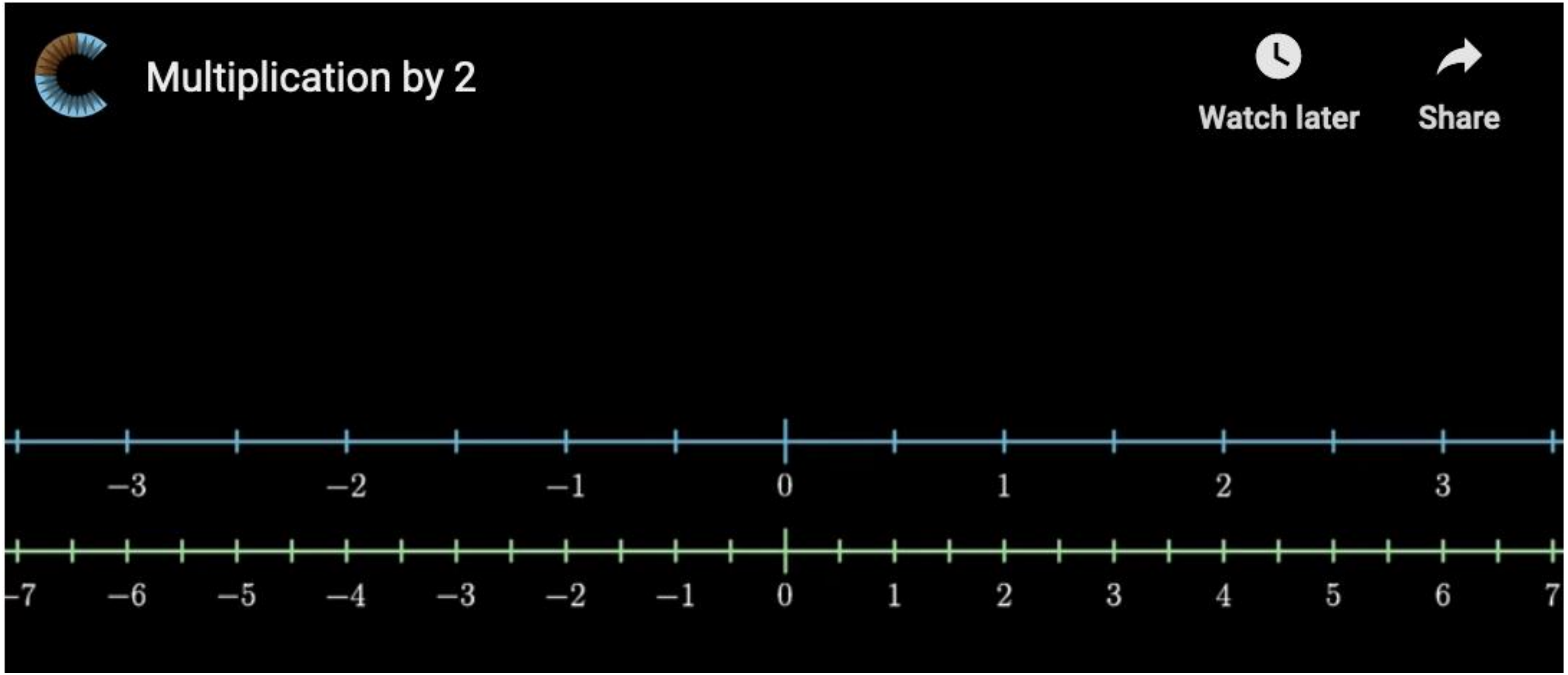
Vector representation of $-3 \cdot (1.5) = -4.5$

- A stretch by a factor of 3
- A reflection about 0 (direction reversal)



A scalar multiple of the number line itself

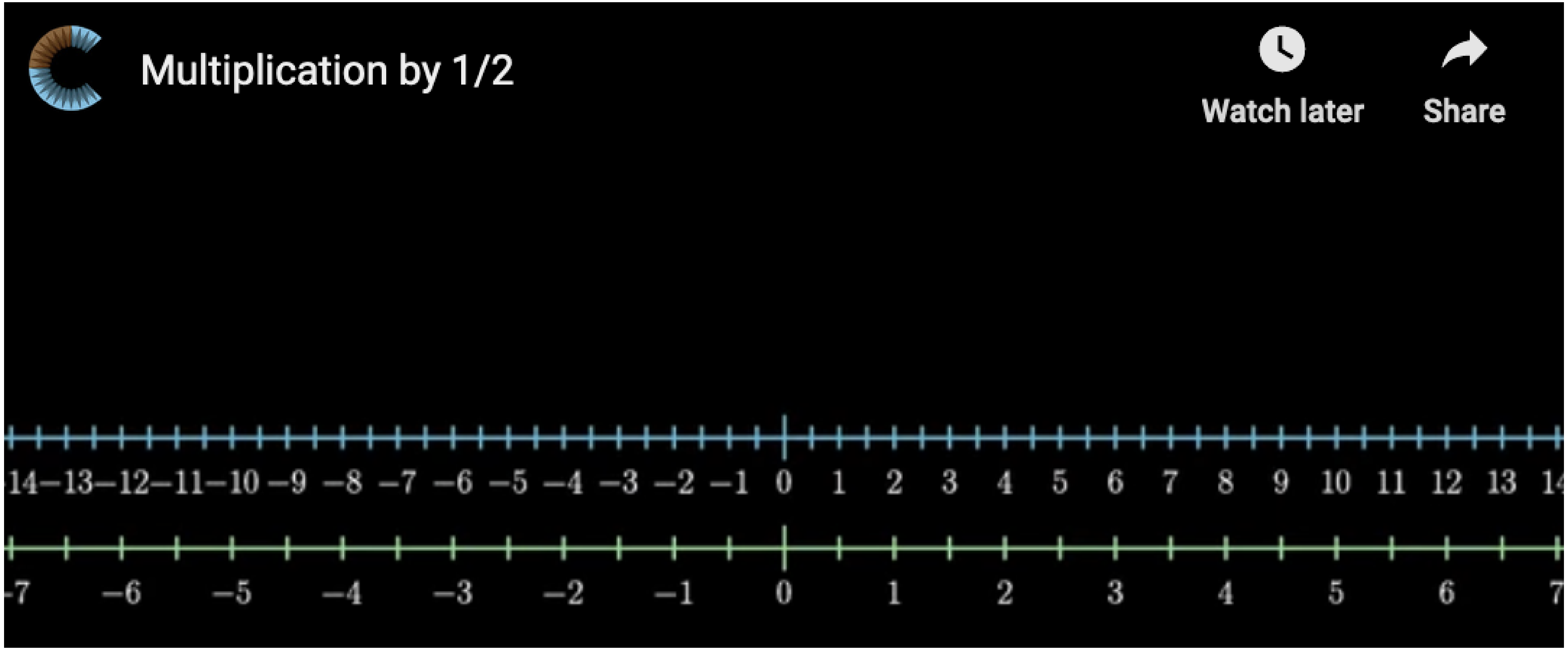
Stretch



Images from
Grant Sanderson (3Blue1Brown)
for Khan Academy

A scalar multiple of the number line itself

Shrink



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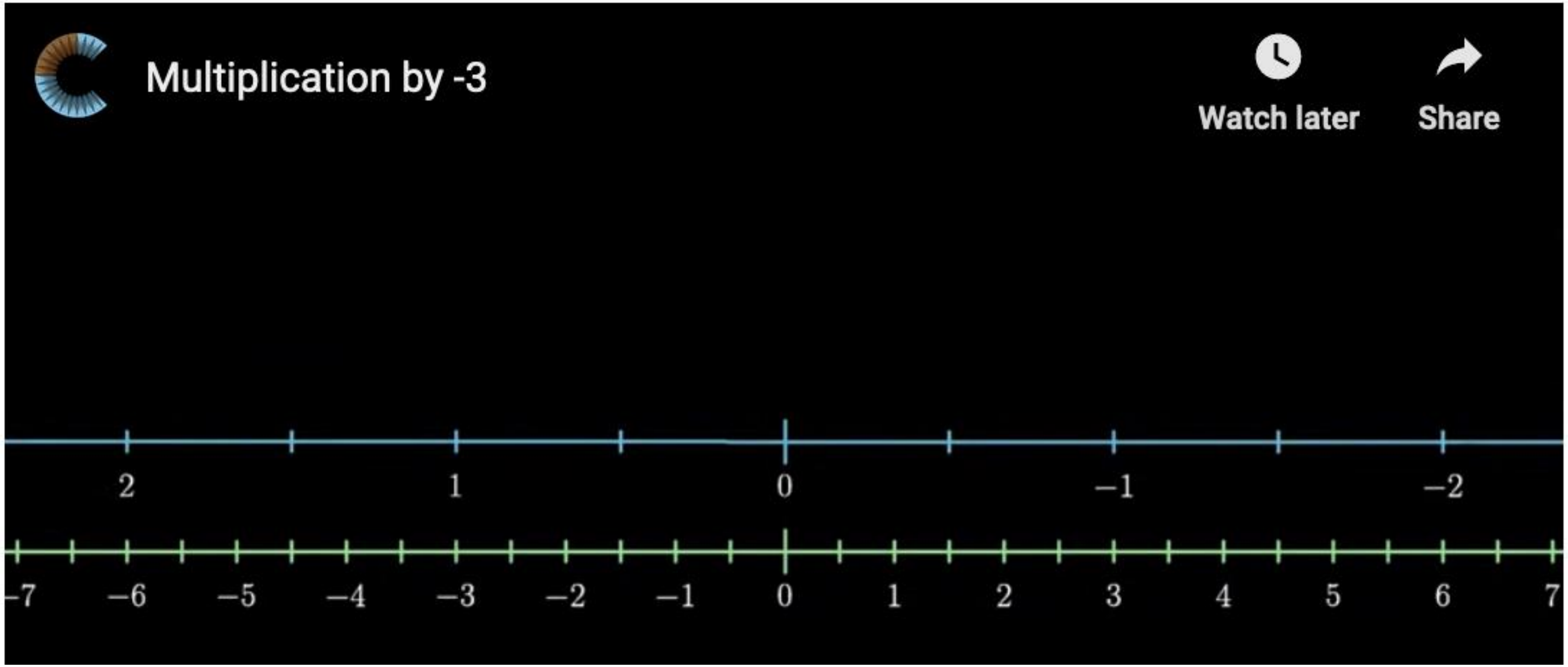
A scalar multiple of the number line itself

Stretch and reflection about the origin



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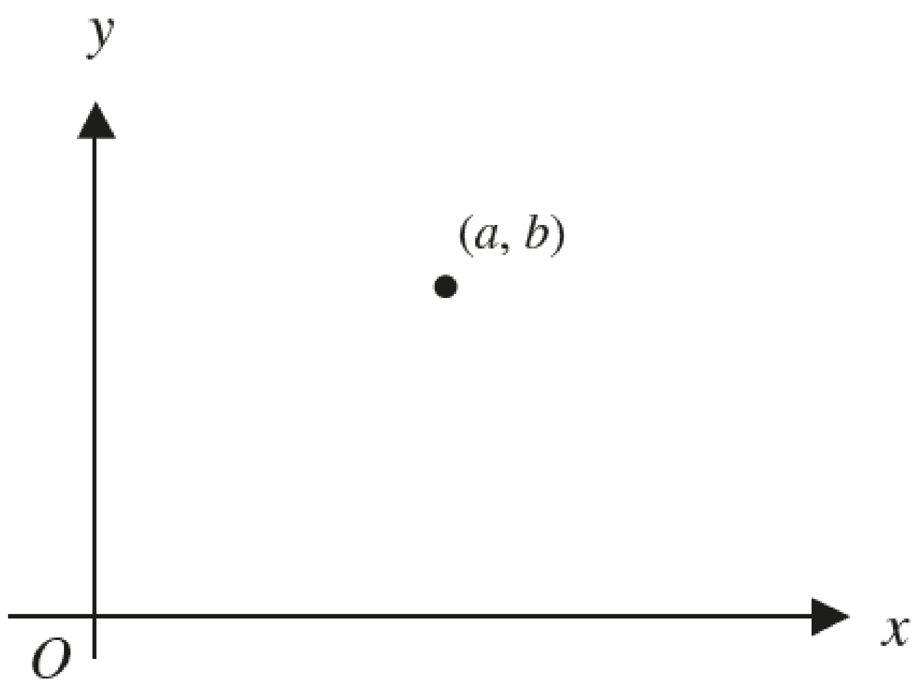
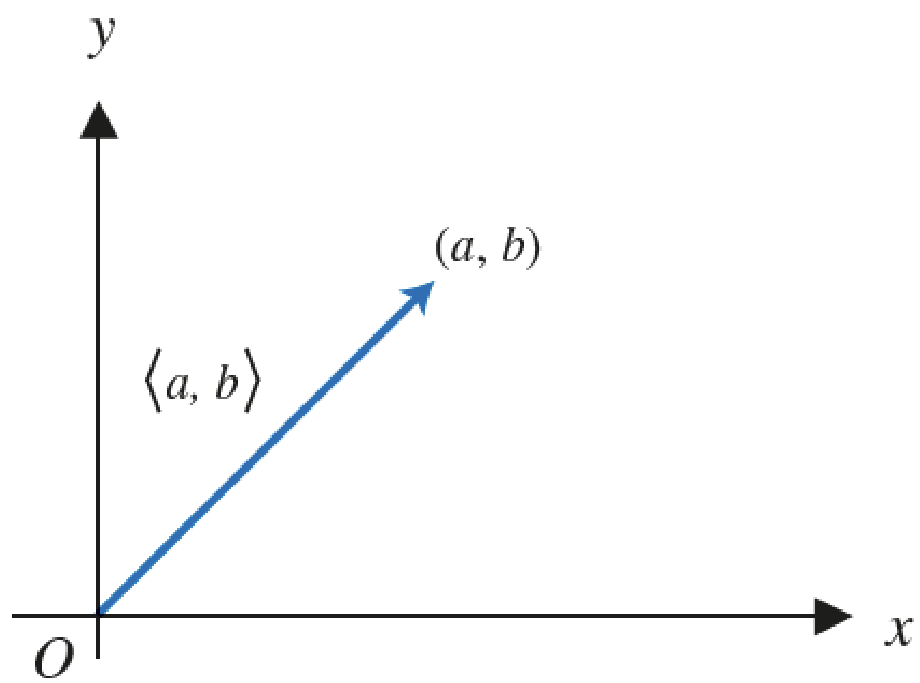
Review of 2 dimensional (2D) concepts

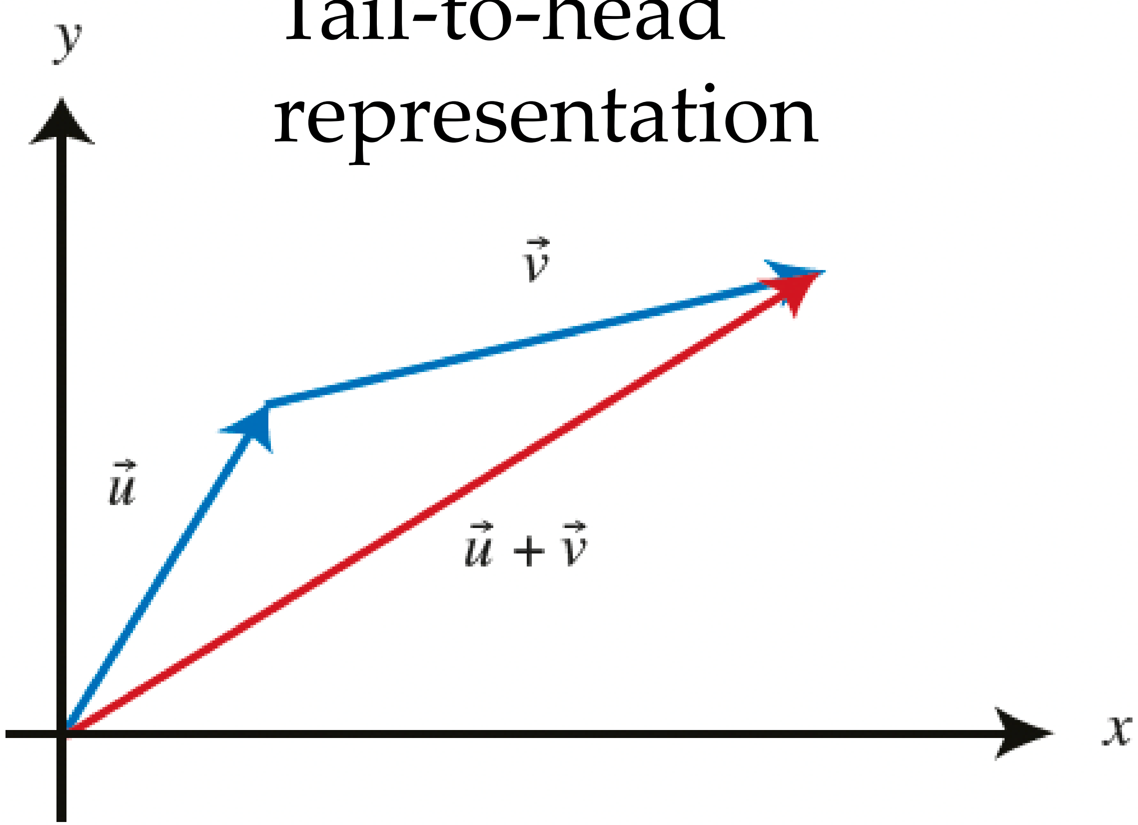
- In 2 dimensions, the **coordinate plane** links an ***ordered pair*** of real numbers (a, b) to a ***point*** in the plane.
- The distance and direction from the origin to a point forms a ***directed line segment***, or **arrow,** which represents a **vector**.



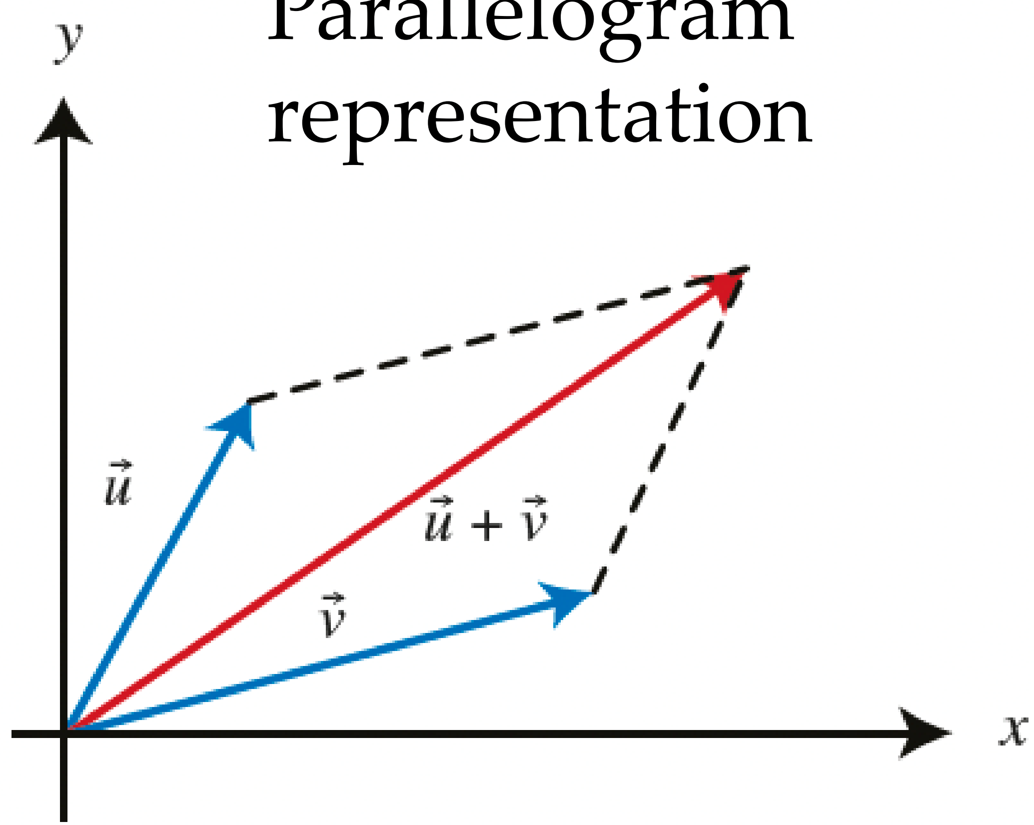


Vector addition in 2 dimensions

Tail-to-head representation

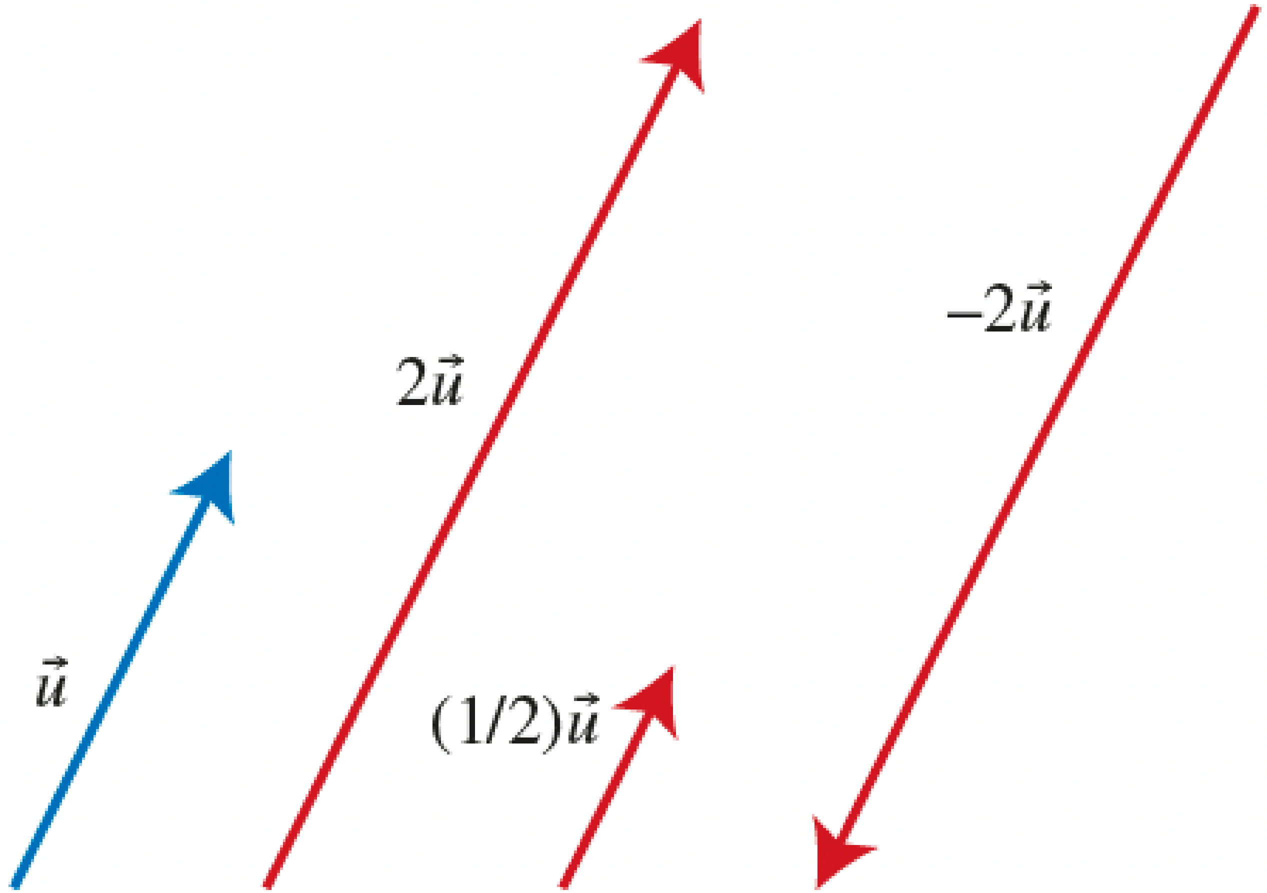


Parallelogram representation



Scalar multiplication in 2 dimensions

Representations of $\vec{u}$ and several scalar multiples of $\vec{u}$.

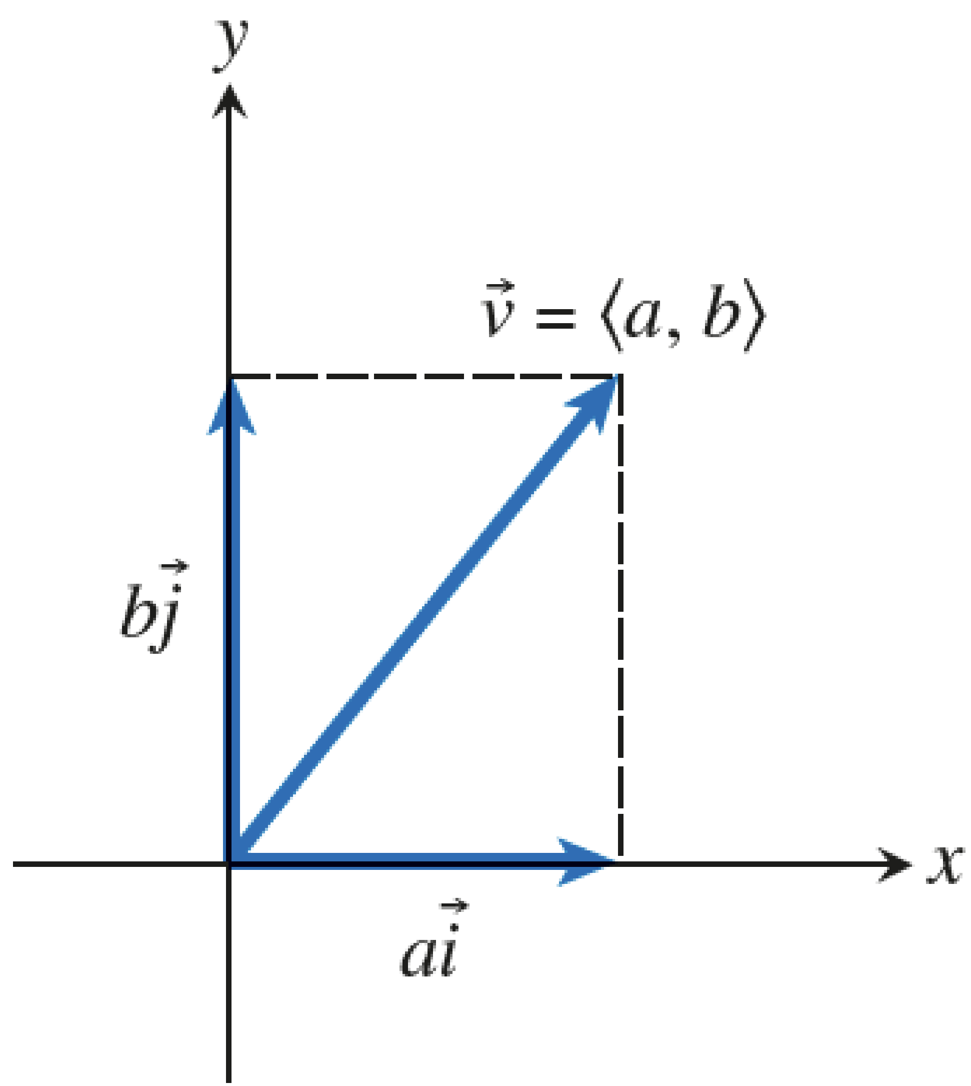


Review of 2 dimensional (2D) concepts

A vector $\langle a, b \rangle$ can be written as a ***linear combination*** of the **standard unit vectors**

$\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$

in the form $a\vec{i} + b\vec{j}$.



Review of key concepts: Vector notation

A vector $\vec{v}$ can be written in 3 ways:

- In component form as $\vec{v} = \langle a, b \rangle$
- In terms of the **standard unit vectors** $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ as $\vec{v} = a\vec{i} + b\vec{j}$
- As a column matrix $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

Students need to *translate* between these forms fluently.

Review of matrix multiplication

To multiply matrices, we multiply *rows* times *columns*:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$$

Thus if $C = \begin{pmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix}$,

then $C \cdot D = \begin{pmatrix} \boxed{1\cdot5+2\cdot8} & 1\cdot6+2\cdot9 & 1\cdot7+2\cdot10 \\ 3\cdot5+4\cdot8 & 3\cdot6+4\cdot9 & 3\cdot7+4\cdot10 \end{pmatrix} = \begin{pmatrix} \boxed{21} & 24 & 27 \\ 47 & 54 & 61 \end{pmatrix}$.

"Cracking the code"

What does **matrix multiplication** do to the ***standard unit vectors***?

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

EXPLORATION 1 **Transforming the Standard Unit Vectors**

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -18 & 24 \\ 37 & -54 \end{bmatrix}$, and $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. Calculate the following matrix products:

(a) $A\vec{i}$ **(b)** $A\vec{j}$ **(c)** $B\vec{i}$ **(d)** $B\vec{j}$ **(e)** $C\vec{i}$ **(f)** $C\vec{j}$

2. What do you notice about these products?

3. How are the products related to the columns of the 2×2 matrices A, B, and C?

"Cracking the code"

What does **matrix multiplication** do to the ***standard unit vectors***?

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Multiplying by $\vec{i}$ captures the 1st column.
- Multiplying by $\vec{j}$ captures the 2nd column.

Thus, to build a 2x2 matrix to perform a *geometric transformation*

1. Think about how $\vec{i}$ and $\vec{j}$ will be transformed.
2. Make these associated image vectors the 1st and 2nd columns of the matrix, respectively.

Experimenting!

bit.ly/NCTMmatrix

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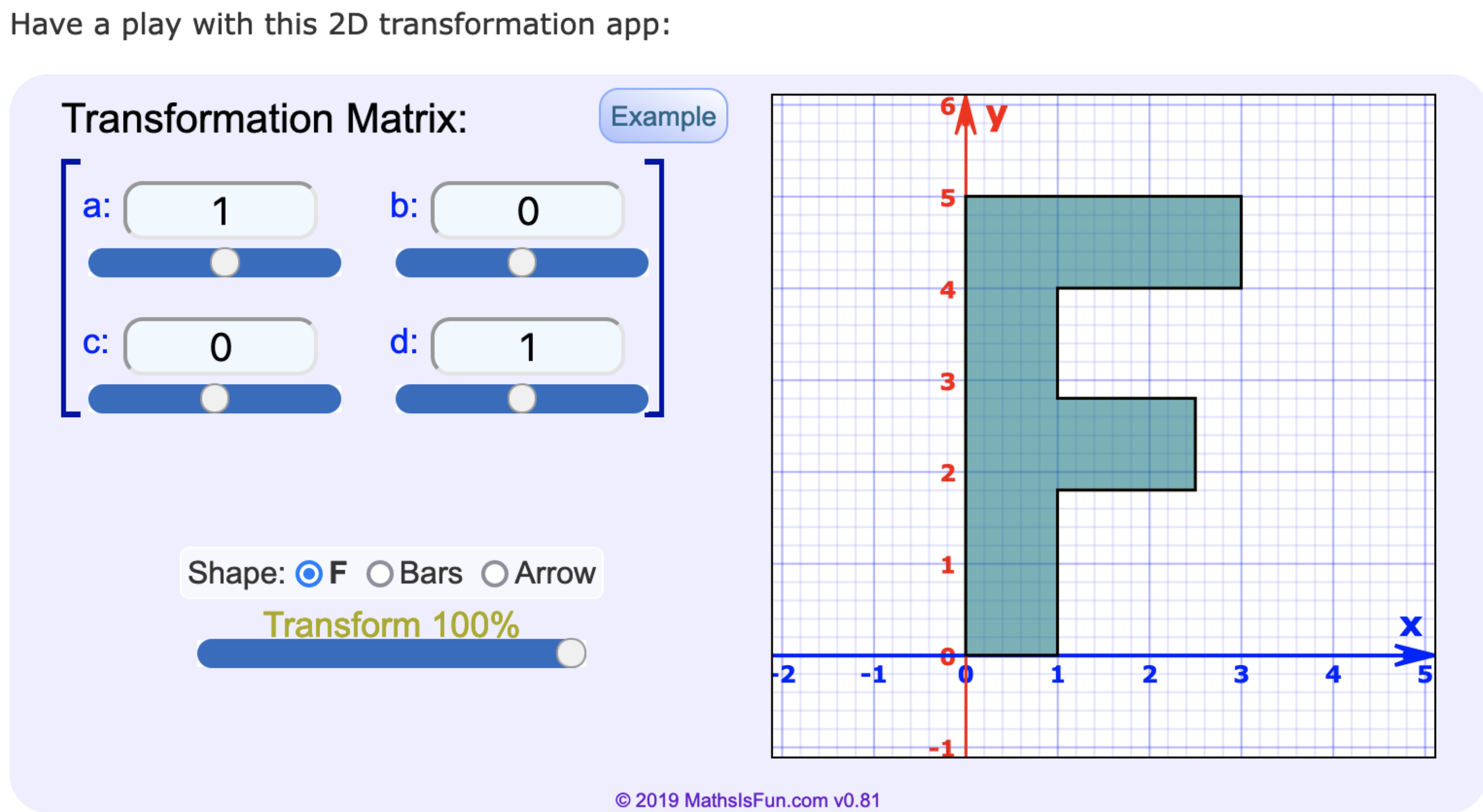


https://www.mathsisfun.com/algebra/matrix-transform.html

Example 1 Reflection

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To perform a reflection over the ***x-axis,*** what does each of the standard unit vectors map to?

- ➢*Construct the transformation matrix*

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

To perform a reflection over the ***y-axis,*** what does each of the standard unit vectors map to?

- ➢*Construct the transformation matrix*

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2 Dilation

Use a *scalar multiple* of the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

DEFINITION Dilation Matrix

The matrix associated with a dilation of the *xy*-plane with respect to the origin by a nonzero factor of k is the **dilation matrix**

$$D_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

Explore using the app!

Example 3 Dilation with unequal factors

What happens if the transformation matrix has different values ($k \neq l$) for the horizontal and vertical dilation factors?

$$D = \begin{bmatrix} k & 0 \\ 0 & l \end{bmatrix}$$

Explore using the app!

Example 4 Rotation

DEFINITION Rotation Matrix

The transformation matrix associated with a counterclockwise rotation about the origin by an angle of θ radians (or degrees) is the **rotation matrix**

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Try building the matrix for a counterclockwise rotation of $\theta =$

1. $\pi/2$
2. π
3. $3\pi/2$

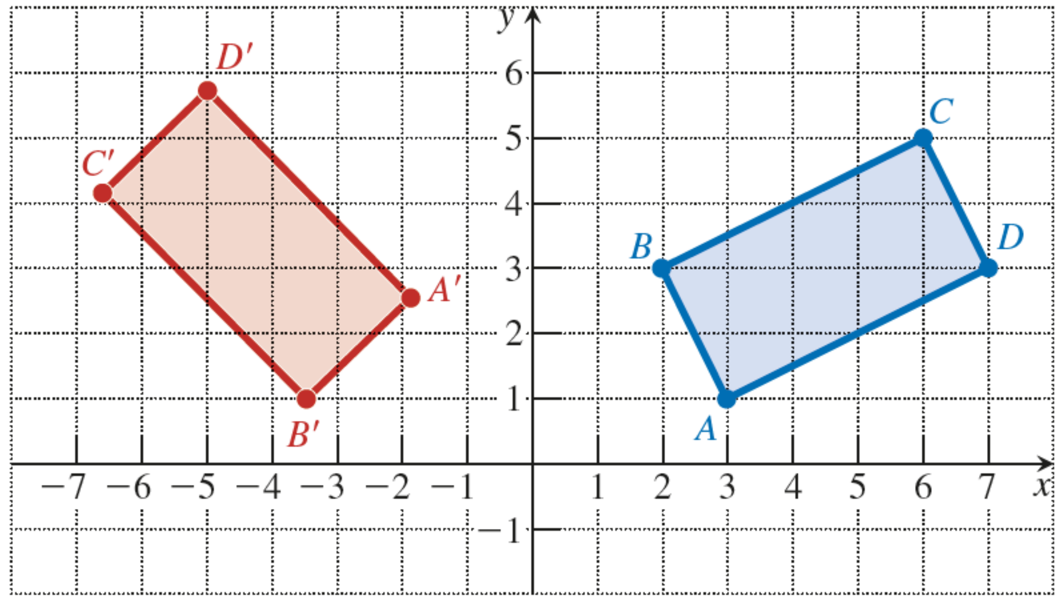
Example 5 Rotating a rectangle

If we write a rectangle as a ***matrix Q of its vertices***, will a rotation matrix still work?

$$Q = \begin{bmatrix} 3 & 2 & 6 & 7 \\ 1 & 3 & 5 & 3 \end{bmatrix}$$

HINT:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



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Example 6 Rotating a conic section

Write the hyperbola $\dfrac{(x-4)^2}{4} - \dfrac{y^2}{9} = 1$ in parametric form and rotate it $\pi/2$ radians counterclockwise

SHOW rotation matrix:

$$\begin{bmatrix} 4 + 2\sec t \\ 3\tan t \end{bmatrix}$$

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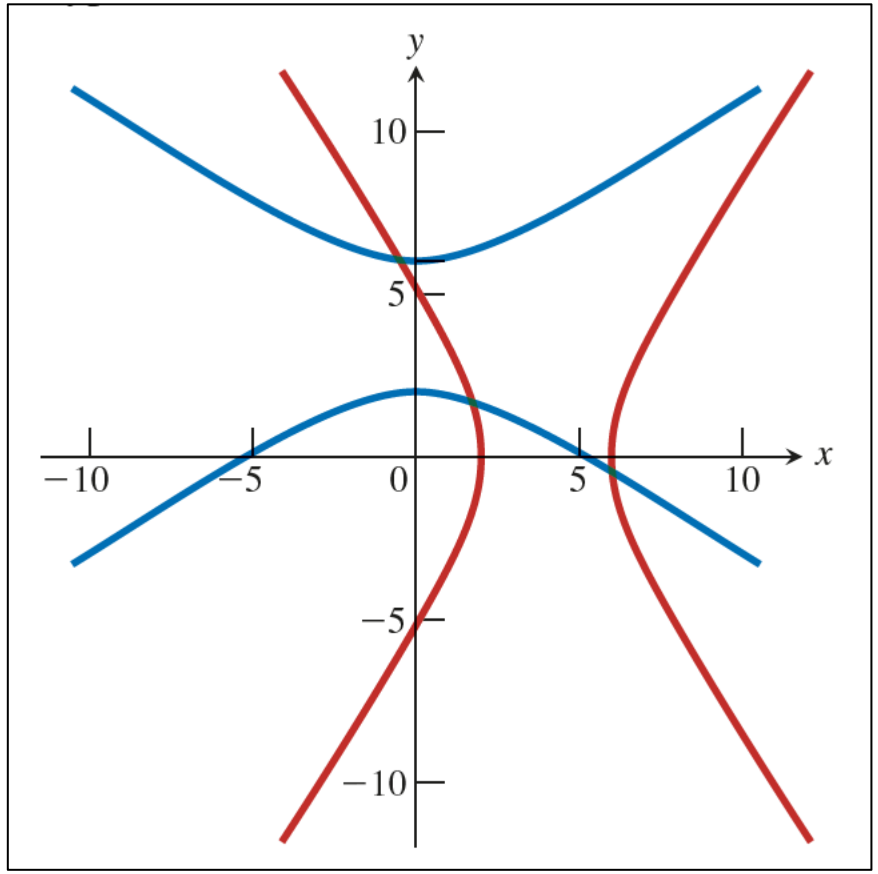


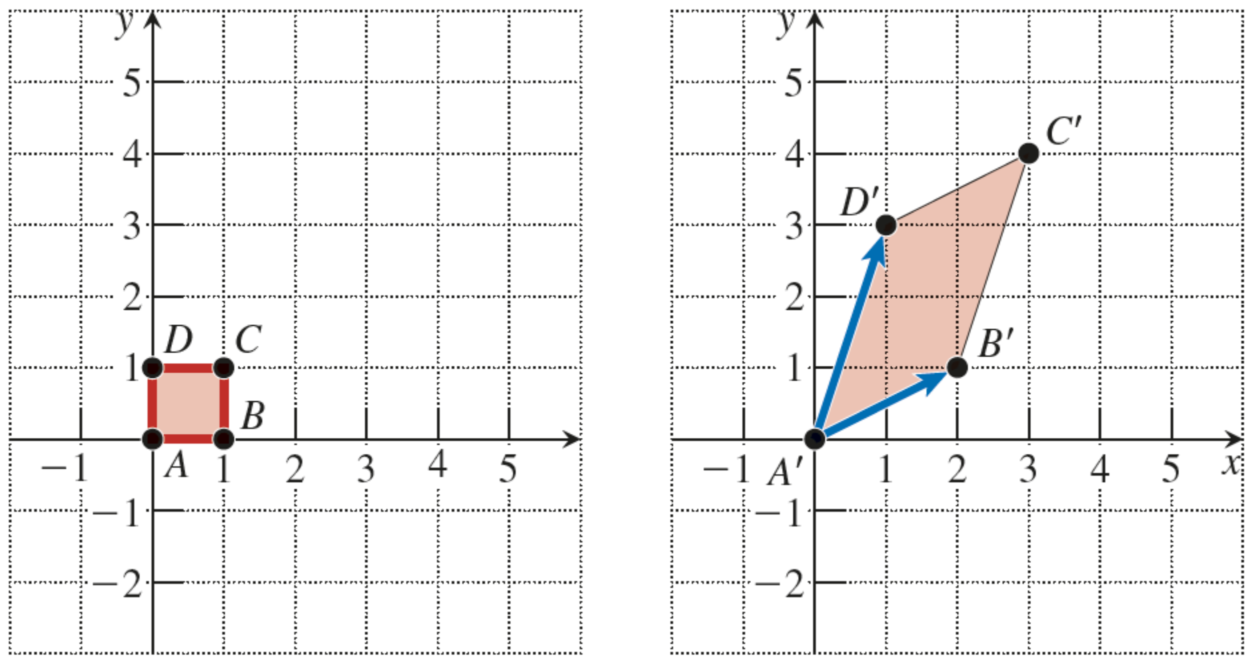
Example 7 Transforming *xy*-plane itself

Given the transformation matrix

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$

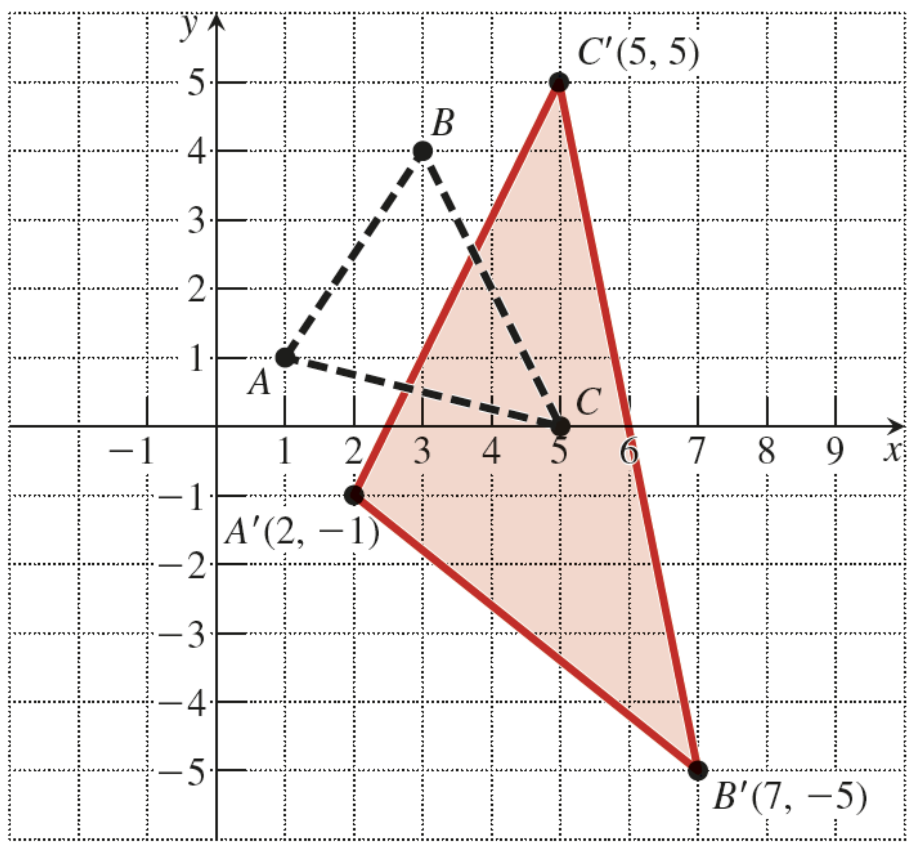
describe the effects of the associated linear transformation on the *xy*-plane.

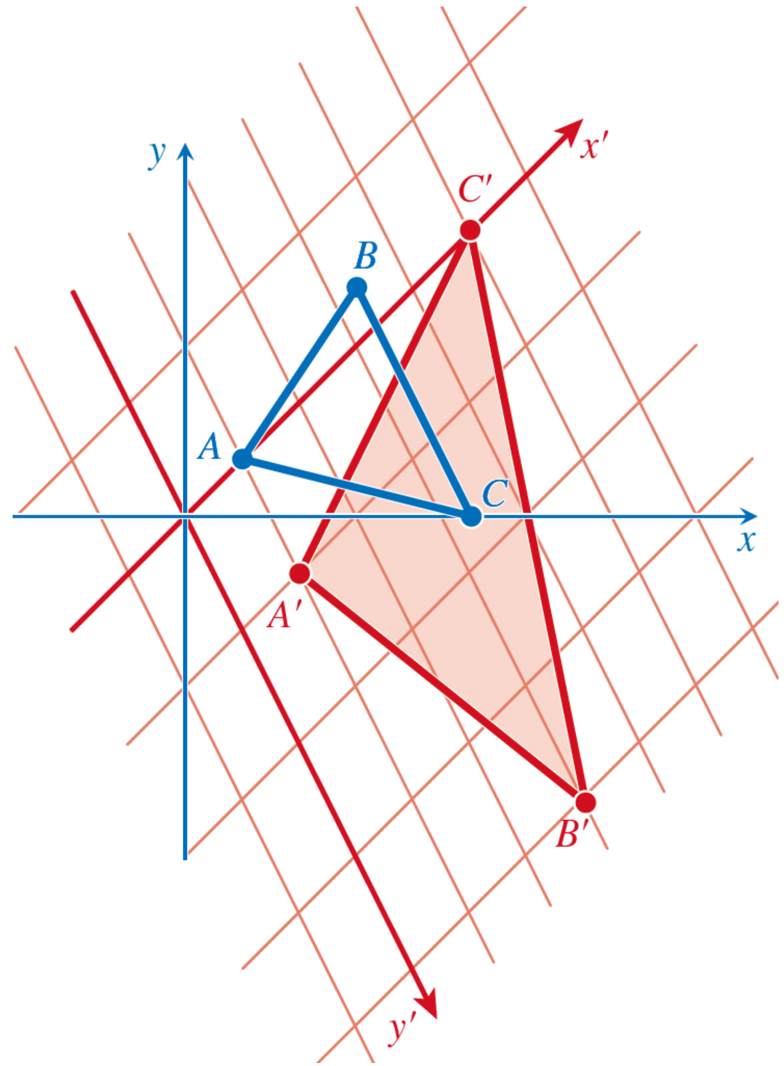
Lines map to lines, and squares map to parallelograms.



Area of $ABCD = 1$ Area of $A'B'C'D' = 5$

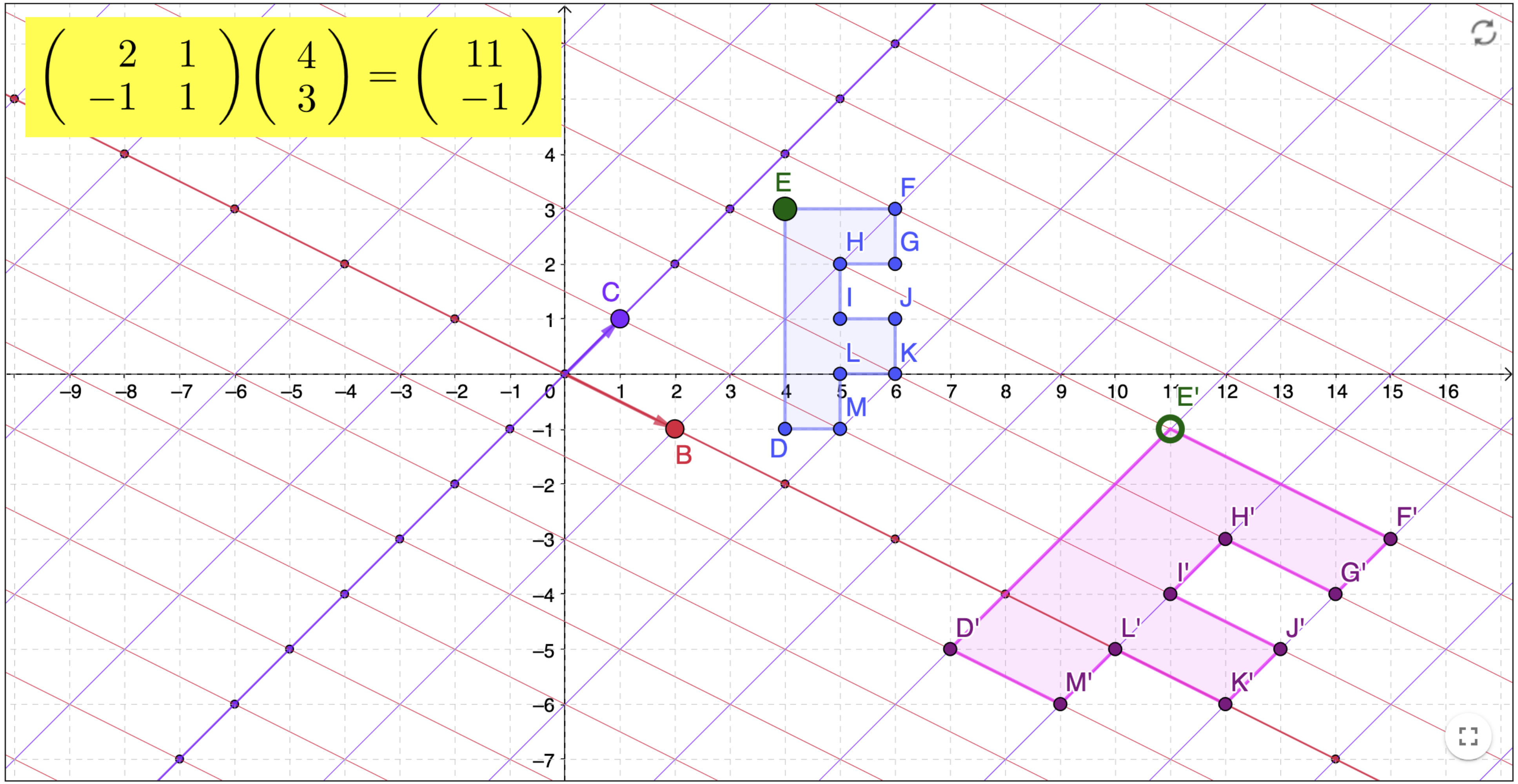
Example 8 Revisiting a transformation





Recap Linear transformations of the *xy*-plane

Lines map to lines, and squares map to parallelograms.



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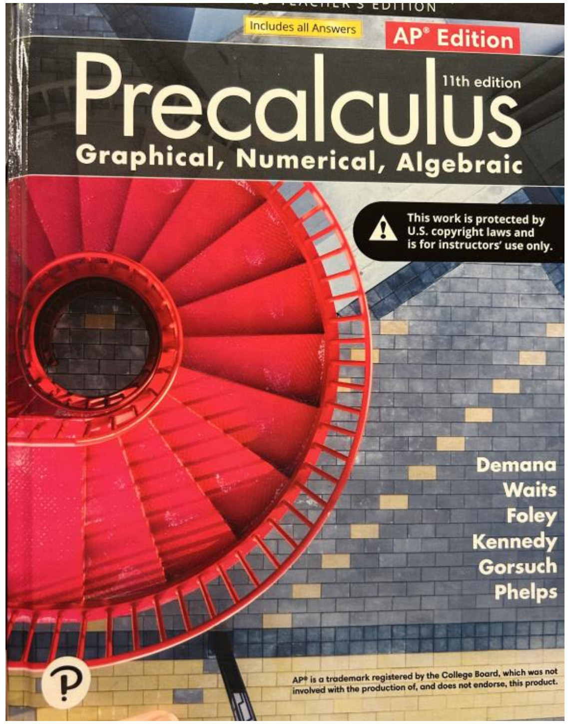
Resources for Matrix Transformations



AP® Edition of Precalculus

Linear transformations and matrices (Grant Sanderson 3Blue1Brown)
https://www.youtube.com/watch?v=kYB8IZa5AuE

Matrices as transformations (Khan Academy article)
https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:matrices/x9e81a4f98389efdf:matrices-as-transformations/a/matrices-as-transformations

Experimental Playground Applets:
https://www.mathsisfun.com/algebra/matrix-transform.html
https://www.geogebra.org/m/nhHhrdQY (Nicola Trubridge)

TI-Nspire Activities/MathNspired
Algebra 2 > Matrices > Matrix Transformations
Precalculus > Matrices > Linear Transformations

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